ICIC Express Letters
Part B: Applications
Volume **14**, Number **4**, April **2023**

ICIC International ©2023 ISSN 2185-2766
pp. 379–387

ALTERNATIVE KALMAN SMOOTHING FILTER WITH MOVING WINDOW STRATEGY

Pyung Soo Kim

System Software Solution Lab.
Department of Electronic Engineering
Tech University of Korea
237 Sangidaehak-ro, Siheung-si, Gyeonggi-do 15073, Korea
pskim@tukorea.ac.kr

Received September 2022; accepted December 2022

Abstract. *In this paper, an alternative Kalman smoothing filter is proposed for discrete-time state-space model. The proposed Kalman smoothing filter is derived from the well-known standard Kalman filter with a moving window strategy using only finite measurements and inputs on the most recent window. The proposed Kalman smoothing filter obtains a posteriori knowledge about the window initial condition from the most recent finite measurements and inputs. The discussion about choosing window length is given. Computer simulation results for a noisy electric motor system validate the effectiveness of the proposed Kalman smoothing filter.*

Keywords: Kalman filter, Kalman smoothing filter, Moving window strategy, Temporary uncertainty, Robustness

1. **Introduction.** The Kalman filter has been recently used in diverse engineering areas for removing noise from a contaminated signal to help reveal important signal features and components [1-6]. Meanwhile, because the Kalman filter is a causal filter providing estimates for states at given times based only on the relative past, the estimates exhibit a delay. Hence, the Kalman smoothing filter has been developed for estimation problems where there is a fixed delay between a measurement and the availability of its estimate [7-10]. Kalman smoothing filters have their own unique features and thus show the following common advantages [7-10]. The smoothing filter generally utilizes more measurement information than the filter to provide state estimates, which can give more accurate estimation performance than the filter. In addition, since the smoothing filter provides state estimates at the delayed time using measurement information up to the current time, measurement information can be reflected in advance in the presence of the state change, which can give more fast convergence than the filter.

However, due to their recursive formulations and infinite memory structure, the Kalman smoothing filter may exhibit performance degradation and even divergence in severe cases for mismodeling and temporary uncertainties. In the case of the standard Kalman filter to resolve this problem, a moving window strategy has been applied successfully [11-14]. Thus, smoothing filters with a moving window strategy have recently been studied [15, 16]. However, these smoothing filters have a serious drawback that the window initial condition has to be handled because the window of past measurements moves forward in time at each sampling time when a new measurement is available. Thus, the smoothing filter requires a posteriori knowledge about the window initial condition as well as finite measurements on the most recent window for each moving window formulation. Since the window initial state is also a state variable and thus not measurable, it is somewhat unreasonable in practical systems that *a posteriori* knowledge about the window initial

DOI: 10.24507/icicelb.14.04.379

condition is assumed to be completely known. Therefore, how to handle the window initial condition might be a challenging issue in the smoothing filter with a moving window strategy. The window initial condition can be assumed to be unknown [15, 16]. However, this assumption might be somewhat heuristic and seems to have no physical meaning.

Therefore, in this paper, an alternative Kalman smoothing filter is proposed for discrete-time state-space model. The proposed Kalman smoothing filter is derived from the well-known standard Kalman filter with a moving window strategy using only finite measurements and inputs on the most recent window. The proposed Kalman smoothing filter obtains *a posteriori* knowledge about the window initial condition from the most recent finite measurements and inputs. The discussion about choosing window length is given. Computer simulations are performed for a noisy electric motor system to verify the proposed Kalman smoother and compare with the Kalman filter with the moving window strategy as well as the standard Kalman smoothing filter. Through computer simulation works, it is shown that the proposed Kalman smoothing filter works well for the nominal system as well as the temporarily uncertain system. It is also shown that the proposed smoother can be remarkably better than two filters for the temporarily uncertain system.

This paper has the following structure. In Section 2, a discrete-time state-space model and a standard Kalman filter are described. In Section 3, an alternative Kalman smoothing filter is developed. In Section 4, the discussion about choosing window length is given. In Section 5, computer simulations are performed. Then, concluding remarks are given in Section 6.

2. **Discrete-Time State-Space Model and Standard Kalman Filter.** A linear discrete-time state-space model with a control input is represented by

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i + Gw_i, \\ z_i &= Cx_i + v_i, \end{aligned} \tag{1}$$

where $x_i \in \Re^n$ is the unknown state, $u_i \in \Re^p$ is the control input, and $z_i \in \Re^q$ is the known measurement. Matrices A, B, C and G represent a system matrix, a control input matrix, a measurement matrix, and a noise matrix, respectively. At the initial time i_0 of system, the state x_{i_0} is a random variable with a mean $\bar{x}_{i_0}$ and a covariance Σ_{i_0}. The system noise $w_i \in \Re^p$ and the measurement noise $v_i \in \Re^q$ are zero-mean white Gaussian and mutually uncorrelated. The covariances of w_i and v_i are denoted by positive definite matrices Q and R, respectively.

The well-known standard Kalman filter [1-4] provides a minimum variance state estimate $\hat{x}_i$, called the one-step predicted estimate of the system state x_i with a control input as follows:

$$\begin{aligned} \hat{x}_{i+1} &= A\hat{x}_i + \left[A\Sigma_i C^T \left(R + C\Sigma_i C^T\right)^{-1}\right](z_i - C\hat{x}_i) + Bu_i \\ &= A\left(I + \Sigma_i C^T R^{-1} C\right)^{-1}\left(\hat{x}_i + \Sigma_i C^T R^{-1} z_i\right) + Bu_i, \end{aligned} \tag{2}$$

$$\begin{aligned} \Sigma_{i+1} &= A\Sigma_i A^T + GQG^T - A\Sigma_i C^T \left(R + C\Sigma_i C^T\right)^{-1} C\Sigma_i A^T \\ &= A\left(I + \Sigma_i C^T R^{-1} C\right)^{-1}\Sigma_i A^T + GQG^T, \end{aligned} \tag{3}$$

where $\hat{x}_{i_0} = \bar{x}_{i_0}$ and Σ_i is the error covariance of the estimate $\hat{x}_i$ with initial value Σ_{i_0}. The Kalman filter has been a standard choice for the state estimation and thus a beautiful reference for diverse engineering areas. The Kalman filter has the recursive formulation for computational efficiency. However, since the Kalman filter processes all past measurements, it tends to accumulate estimation errors during its implementation. Therefore, the Kalman filter has been known to show performance degradation and even divergence phenomena for mismodeling and temporary uncertainties.

3. **Alternative Kalman Smoothing Filter with Moving Window Strategy as well as Handling Window Initial Condition.** From the standard Kalman filter (2) and (3), an alternative Kalman smoothing filter is developed to estimate the state x_{i-d} at the lagged time $i-d$. The lagged time $i-d$ means there is a fixed delay between the measurement and the availability of its estimate. The positive integer d is the delay length satisfying $0 \le d < M$ and equal to the number of discrete time steps between the lagged time $i-d$ at which the state is to be estimated and the current time i of the last measurement used in estimating it.

To apply the moving window strategy, only finite measurements as well as inputs on the most recent window $\left[i - M \left(\triangleq i_M\right), i\right]$ are utilized for smoothing process. Hence, the standard Kalman filter (2) and (3) can be modified by $\hat{x}_{i_M+j}$ on the window $[i_M, i]$ as follows:

$$\hat{x}_{i_M+j+1} = A\left(I + \Sigma_{i_M+j}C^T R^{-1} C\right)^{-1}\left(\hat{x}_{i_M+j} + \Sigma_{i_M+j}C^T R^{-1} z_{i_M+j}\right) + Bu_{i_M+j}, \tag{4}$$

where the error covariance Σ_{i_M+j} is given by

$$\Sigma_{i_M+j+1} = A\left(I + \Sigma_{i_M+j}C^T R^{-1} C\right)^{-1}\Sigma_{i_M+j}A^T + GQG^T, \tag{5}$$

and $0 \le j \le M-1$. Then, at the current time i, the state estimate $\hat{x}_i$ can be represented by the following form:

$$\hat{x}_i = \Phi_M \hat{x}_{i_M} + \sum_{j=0}^{M-1} \Phi_{M-j}\Sigma_{i_M+j}C^T R^{-1} z_{i_M+j} + \sum_{j=0}^{M-1}\Phi_{M-j}Bu_{i_M+j},$$

where the matrix Φ_j is given by

$$\Phi_{j+1} = \Phi_j A\left[I + \Sigma_{i_M+M-j-1}C^T R^{-1} C\right]^{-1}, \ \Phi_0 = I.$$

Therefore, at the lagged time $i-d$, the alternative Kalman smoothing filter $\hat{x}_{i-d}$ can be represented by the summation form with the window initial condition as follows:

$$\hat{x}_{i-d} = \Phi_{M-d}\hat{x}_{i_M} + \sum_{j=0}^{M-d-1}\Phi_{M-j}\Sigma_{i_M+j}C^T R^{-1} z_{i_M+j} + \sum_{j=0}^{M-d-1}\Phi_{M-j}Bu_{i_M+j}. \tag{6}$$

At this point, *a posteriori* knowledge about the window initial condition $\{\hat{x}_{i_M}, \Sigma_{i_M}\}$ on the window $[i_M, i]$ is required for (5) and (6).

With the window initial state x_{i_M}, finite measurements Z_i and inputs U_i on the most recent window $[i_M, i]$ can be expressed by the following regression form

$$Z_i - \Xi U_i = \Gamma x_{i_M} + \Lambda W_i + V_i, \tag{7}$$

where Z_i and U_i are defined by

$$Z_i \triangleq \begin{bmatrix} z_{i_M} \\ z_{i_M+1} \\ \vdots \\ z_{i-2} \\ z_{i-1} \end{bmatrix}, \quad U_i \triangleq \begin{bmatrix} u_{i_M} \\ u_{i_M+1} \\ \vdots \\ u_{i-2} \\ u_{i-1} \end{bmatrix}, \tag{8}$$

and W_i, V_i have the same form as (8) for w_i, v_i, respectively. Matrices Γ, Ξ and Λ are defined by

$$\Gamma \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{M-2} \\ CA^{M-1} \end{bmatrix}, \quad \Xi \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ CB & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CA^{M-3}B & CA^{M-4}B & \cdots & 0 & 0 \\ CA^{M-2}B & CA^{M-3}B & \cdots & CB & 0 \end{bmatrix},$$

$$\Lambda \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ CG & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CA^{M-3}G & CA^{M-4}G & \cdots & 0 & 0 \\ CA^{M-2}G & CA^{M-3}G & \cdots & CG & 0 \end{bmatrix}. \tag{9}$$

The noise term $\Lambda W_i + V_i$ in (7) is zero-mean white Gaussian as follows:

$$\Lambda W_i + V_i \sim \mathcal{N}\left(Z_i - \Xi U_i; 0, \Pi\right), \tag{10}$$

where $\mathcal{N}\left(Z_i - \Xi U_i; 0, \Pi\right)$ denotes the Gaussian probability density function evaluated at $Z_i - \Xi U_i$ with zero-mean and covariance matrix

$$\Pi \triangleq \Lambda \left[\operatorname{diag}\left(\overbrace{Q\ Q\ Q\ \cdots\ Q}^{M}\right)\right]\Lambda^T + \left[\operatorname{diag}\left(\overbrace{R\ R\ R\ \cdots\ R}^{M}\right)\right], \tag{11}$$

where diag$(Q\ Q\ Q\ \cdots\ Q)$ and diag$(R\ R\ R\ \cdots\ R)$ denote block-diagonal matrices with M elements of Q and R, respectively.

Then, using the approach of best linear unbiased estimation in [17], the window initial condition $\hat{x}_{i_M}$ is obtained by

$$\hat{x}_{i_M} = \left(\Gamma^T \Pi^{-1} \Gamma\right)^{-1} \Gamma^T \Pi^{-1} \left(Z_i - \Xi U_i\right). \tag{12}$$

In addition, the window initial condition Σ_{i_M} is obtained by the error covariance of $\hat{x}_{i_M}$ as follows:

$$\begin{aligned} &\Sigma_{i_M} \\ &= \mathbf{E}\left[\left\{x_{i_M} - \hat{x}_{i_M}\right\}\left\{x_{i_M} - \hat{x}_{i_M}\right\}^T\right] \\ &= \mathbf{E}\left[\left\{x_{i_M} - \left(\Gamma^T\Pi^{-1}\Gamma\right)^{-1}\Gamma^T\Pi^{-1}\left(Z_i - \Xi U_i\right)\right\}\left\{x_{i_M} - \left(\Gamma^T\Pi^{-1}\Gamma\right)^{-1}\Gamma^T\Pi^{-1}\left(Z_i - \Xi U_i\right)\right\}^T\right] \\ &= \left(\Gamma^T\Pi^{-1}\Gamma\right)^{-1}. \end{aligned} \tag{13}$$

Therefore, *a posteriori* knowledge about the window initial condition $\{\hat{x}_{i_M}, \Sigma_{i_M}\}$ in (12) and (13) on the window $[i_M, i]$ is given for (5) and (6) in the *unbiasedness* sense. As shown in (13), the window initial condition Σ_{i_M} is constant value. Thus, the error covariance Σ_{i_M+j} (5) defined on the window $[i_M, i]$ can be rewritten as follows:

$$\Sigma_{j+1} = A\left(I + \Sigma_j C^T R^{-1} C\right)^{-1} \Sigma_j A^T + GQG^T,\ 0 \le j \le M-1, \tag{14}$$

with the window initial condition $\Sigma_0 = \Sigma_{i_M} = \left(\Gamma^T\Pi^{-1}\Gamma\right)^{-1}$.

Using (12), the Kalman smoothing filter (6) with the window initial condition $\hat{x}_{i_M}$ can be rewritten by

$$\begin{aligned} \hat{x}_{i-d} &= \Phi_{M-d}\hat{x}_{i_M} + \sum_{j=0}^{M-d-1} \Phi_{M-j}\Sigma_j C^T R^{-1} z_{i_M+j} + \sum_{j=0}^{M-d-1} \Phi_{M-j} B u_{i_M+j} \\ &= \Phi_{M-d}\left(\Gamma^T\Pi^{-1}\Gamma\right)^{-1}\bar{\Gamma}^T\bar{\Pi}^{-1}\left(Z_i - \Xi U_i\right) + \sum_{j=0}^{M-d-1} \Phi_{M-j}\Sigma_j C^T R^{-1} z_{i_M+j} \\ &\quad + \sum_{j=0}^{M-d-1} \Phi_{M-j} B u_{i_M+j} \\ &= \Phi_{M-d}\Sigma_0\Gamma^T\Pi^{-1}\left(Z_i - \Xi U_i\right) + \sum_{j=0}^{M-d-1} \Phi_{M-j}\Sigma_j C^T R^{-1} z_{i_M+j} + \sum_{j=0}^{M-d-1} \Phi_{M-j} B u_{i_M+j} \end{aligned}$$

$$= \Phi_{M-d}\Sigma_0\Gamma^T\Pi^{-1}Z_i + \left[\overbrace{\Phi_M\Sigma_0 \ \Phi_{M-1}\Sigma_1 \ \cdots \ \Phi_{d+1}\Sigma_{M-d-1}}^{M-d} \ \overbrace{0 \ 0 \ \cdots \ 0}^{d}\right] C^T R^{-1} Z_i$$

$$- \Phi_{M-d}\Sigma_0\Gamma^T\Pi^{-1}\Xi U_i + \left[\overbrace{\Phi_M \ \Phi_{M-1} \ \cdots \ \Phi_{d+1}}^{M-d} \ \overbrace{0 \ 0 \ \cdots \ 0}^{d}\right] BU_i, \tag{15}$$

where the transition matrix Φ_j is given by

$$\Phi_{j+1} = \Phi_j A \left[I + \Sigma_{M-j-1} C^T R^{-1} C\right]^{-1}, \ \Phi_0 = I. \tag{16}$$

Finally, the proposed Kalman smoothing filter $\hat{x}_{i-d}$ (15) can be expressed by the simple matrix form as the following theorem.

Theorem 3.1. *Assume that $\{A, C\}$ is observable and $M \geq n$. Then, the proposed Kalman smoothing filter $\hat{x}_{i-d}$ on the window $[i-M, i]$ is expressed by the following simple matrix form:*

$$\hat{x}_{i-d} \stackrel{\triangle}{=} (\mathcal{H}_{\mathcal{I}} + \mathcal{H}_{\mathcal{Z}}) Z_i + (-\mathcal{H}_{\mathcal{I}}\Xi + \mathcal{H}_{\mathcal{U}}) U_i, \tag{17}$$

where matrices $\mathcal{H}_{\mathcal{I}}$, $\mathcal{H}_{\mathcal{Z}}$, and $\mathcal{H}_{\mathcal{U}}$ are as follows:

$$\mathcal{H}_{\mathcal{I}} = \Phi_{M-d}\Sigma_0\Gamma^T\Pi^{-1}, \tag{18}$$

$$\mathcal{H}_{\mathcal{Z}} = \left[\overbrace{\Phi_M\Sigma_0 \ \Phi_{M-1}\Sigma_1 \ \cdots \ \Phi_{d+1}\Sigma_{M-d-1}}^{M-d} \ \overbrace{0 \ 0 \ \cdots \ 0}^{d}\right] C^T R^{-1}, \tag{19}$$

$$\mathcal{H}_{\mathcal{U}} = \left[\overbrace{\Phi_M \ \Phi_{M-1} \ \cdots \ \Phi_{d+1}}^{M-d} \ \overbrace{0 \ 0 \ \cdots \ 0}^{d}\right] B. \tag{20}$$

It is noted that matrices (18), (19), and (20) require computation only on the interval $[0, M]$ once and is time-invariant for all windows.

Theorem 3.1 means the proposed Kalman smoothing filter $\hat{x}_{i-d}$ (17) is time-invariant. The on-line computation of the proposed Kalman smoothing filter requires only smoothing updates. Hence, the computational complexity of the proposed Kalman smoothing filter is $\mathcal{O}(M)$ and thus linear in the size of the window length M. In practice, this means that quite a large M can be chosen without worrying about computational burden.

4. **Choice of Window Length.** The window length M can be a useful design parameter for the proposed Kalman smoothing filter. Thus, the important issue here is how to choose an appropriate window length M that makes the proposed Kalman smoothing filter's performance as good as possible. The noise suppression of the proposed Kalman smoothing filter might be closely related to the window length M, and it can have greater noise suppression as the window length M increases. That is, choosing a larger M generally results in better noise reduction performance, since more measurements are taken into account. However, at the same time, a larger M increases the convergence time of the proposed Kalman smoothing filtered estimate. In addition, the complexity of computing the Kalman smoothing filter, though this complexity is only $\mathcal{O}(M)$, becomes larger as the window length increases. This illustrates the proposed Kalman smoothing filter's compromise between noise suppression and tracking ability. Therefore, from an engineering perspective, there is tradeoff that chooses the window length M. Since window length M is an integer, fine adjustment of the properties with M is difficult. Moreover, it is difficult to determine the window length systematically. In applications, one method of determining the window length is to take the appropriate value that can provide sufficient noise suppression. A heuristic would be to determine window length M in advance based

on the error covariance Σ_j in (14). When the $L2$ norm of error covariance matrix Σ_M falls below a certain threshold, M in the vicinity is set the window length M. Since the error covariance generally decreases over time, this heuristic allows to choose M in terms of the $L2$ norm of the error covariance. This heuristic will be implemented in computer simulations. Therefore, it can be stated from the above discussions that the window length M can be considered a useful parameter to make the performance of the proposed Kalman smoothing filter as good as possible.

5. **Computer Simulations.** In this section, the proposed Kalman smoothing filter is applied for a direct current (DC) motor system through computer simulations. System matrices for discrete-time state-space model (1) for the DC motor system are as follows [18]:

$$A = \begin{bmatrix} 0.8178 & -0.0011 \\ 0.0563 & 0.3678 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1813 \\ 0.0069 \end{bmatrix}, \quad G = \begin{bmatrix} 0.0006 \\ 0.0057 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \tag{21}$$

where the motor is operated without any payload, and the armature current and the rotational speed are chosen as outputs measured by sensors. The DC motor encounters an external source, that is, the input voltage to drive the motor. This external source is treated as a control input u_i. The control input is emulated by the unit step. For the noisy DC motor system (21), system and measurement noise covariances are taken by $Q = 0.02^2$ and $R = 0.03^2$, respectively.

Even if various dynamic systems and signal systems are represented in state-space model accurately on a long time scale, it may undergo unpredictable changes, such as jumps in frequency, phase, and velocity. Because these effects typically occur over a short time horizon, they are called temporary uncertainties [11-14]. As representative temporary uncertainties, there are a model uncertainty, an unknown input, and incomplete measurement information, etc. The state estimation for dynamic systems should be robust to diminish the effects of these temporary uncertainties. In order to verify intrinsic robustness property of the proposed Kalman smoothing filter, the DC motor system is assumed to have a temporary model uncertainty as follows:

$$\begin{aligned} x_{i+1} &= (A + \Delta A_i)\, x_i + Bu_i + Gw_i, \\ y_i &= (C + \Delta C_i)\, x_i + v_i, \end{aligned} \tag{22}$$

where

$$\Delta A_i = \delta_i \cdot I_{2\times 2}, \quad \Delta C_i = \begin{bmatrix} 0.2\delta_i & 0.2\delta_i \end{bmatrix},$$

with

$$\delta_i = \begin{cases} 0.05 & \text{if } 150 \le i \le 200, \\ 0 & \text{otherwise.} \end{cases}$$

Although the proposed Kalman smoothing filter is designed by the nominal discrete-time state-space model (21), actual measurements for the smoothing are obtained from the temporarily uncertain system (22). To make a clearer comparison of estimation performances, simulations of 30 runs are performed using different noises.

Before actual simulations, the $L2$ norm of error covariance matrix Σ_M is computed from the error covariance equation (14) in order to determine the optimal window length M that can provide enough noise suppression. The $L2$ norm of error covariance matrix Σ_M is obtained from the Matlab function `norm(X)`. This function returns the $L2$ norm or maximum singular value of matrix X, which can be also implemented approximately using another Matlab function `max(svd(X))`. The $L2$ norm of error covariance matrix Σ_M is plotted according to increasing window lengths in Figure 1. It can be seen that the $L2$ norm of error covariance matrix reduces as the window length grows and converges when the window length is around $M = 20$. Of course, the $L2$ norm of error covariance

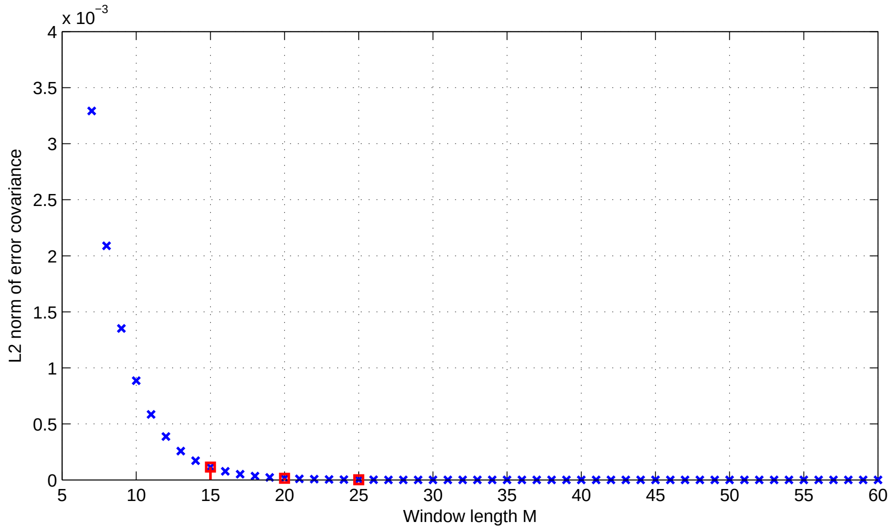

FIGURE 1. Choosing optimal window length using $L2$ norm of Σ_M

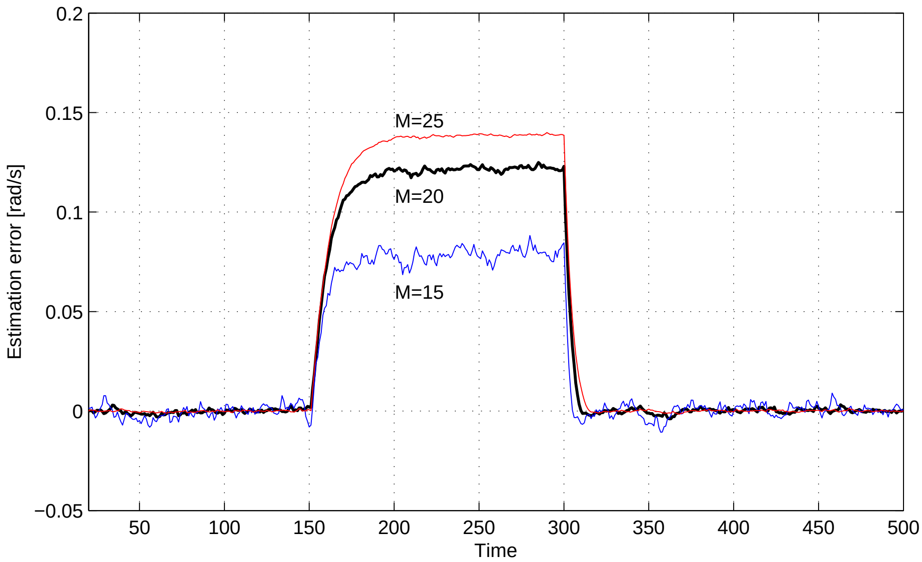

FIGURE 2. Estimation errors of the proposed Kalman smoothing filter according to diverse window lengths

matrix can be more reduced when $M > 20$. For diverse window lengths such as $M = 15$, $M = 20$, and $M = 25$, estimation errors of the proposed Kalman smoothing filter are compared. As mentioned already in Section 4, the proposed Kalman smoothing filter can have greater noise suppression as the window length M increases. On the other hand, at the same time, a larger M increases the convergence time of the proposed Kalman smoothing filtered estimate. Therefore, from an engineering perspective, there is tradeoff that chooses the window length M, which can be confirmed in Figure 2.

To illustrate the validity of the proposed Kalman smoothing filter and to compare with the Kalman filter with the moving window strategy and the standard Kalman smoothing filter, the window length is taken by $M = 20$. Figure 3 shows root-mean-square (RMS) estimation errors for 30 simulations. Figure 4 shows estimation errors for one of 30 simulations. As shown in simulation results, the estimation error of the proposed Kalman smoothing filter is smaller than those of other two filters on the interval where modeling uncertainty exists. In addition, the convergence of estimation error is much faster

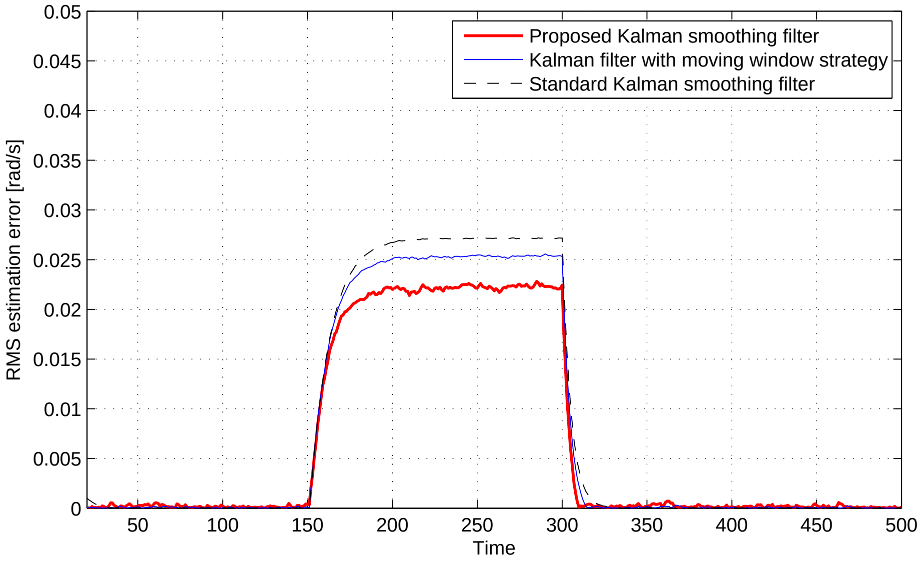

FIGURE 3. Comparison of RMS estimation errors ($M = 20$)

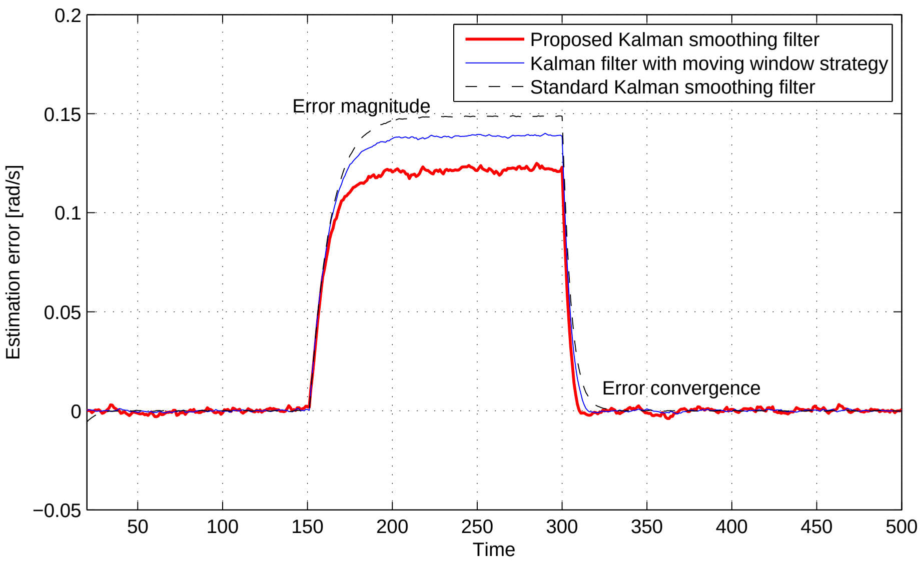

FIGURE 4. Comparison of estimation errors ($M = 20$)

than those of other two filters after temporary uncertainty disappears. Moreover, the proposed Kalman smoothing filter can be comparable to the other two filters after the effect of temporary uncertainty completely disappears. Therefore, the proposed Kalman smoothing filter can be more robust than the other two filters when applied to temporarily uncertain systems, although it is designed with no consideration of robustness.

6. **Conclusion.** An alternative Kalman smoothing filter has been proposed for discrete-time state-space model. The proposed Kalman smoothing filter has been derived from the well known standard Kalman filter with a moving window strategy using only finite measurements and inputs on the most recent window. The discussion about choosing window length has been given. Computer simulations have been performed for a noisy electric motor system to verify the proposed Kalman smoother and compare with the Kalman filter with the moving window strategy as well as the standard Kalman smoothing

filter. Through computer simulation works, it has been shown that the proposed Kalman smoothing filter works well for the nominal system as well as the temporarily uncertain system. It has been also shown that the proposed smoothing filter can be remarkably better than two filters for the temporarily uncertain system.

Time-varying systems can be often used for many practical and real-time applications. Thus, the smoothing filter for time-varying systems is necessary. In addition, in order to improve computational reliability and overcome computational burden, the computational efficiency should be considered for the implementation of the time-varying smoothing filter. These can be considered as future research topics.

Acknowledgment. This research was supported by the MSIT (Ministry of Science and ICT), Korea, under the ICAN (ICT Challenge and Advanced Network of HRD) program (IITP-2022-RS-2022-00156326) supervised by the IITP (Institute of Information & Communications Technology Planning & Evaluation).

REFERENCES

[1] M. Grewal, Applications of Kalman filtering in aerospace 1960 to the present, *IEEE Control Systems Magazine*, vol.30, no.3, pp.69-78, 2010.

[2] F. Auger, M. Hilairet, J. M. Guerrero, E. Monmasson, T. Orlowska-Kowalska and S. Katsura, Industrial applications of the Kalman filter: A review, *IEEE Trans. Industrial Electronics*, vol.60, no.12, pp.5458-5471, 2013.

[3] M. Rhudy, R. Salguero and K. Holappa, A Kalman filtering tutorial for undergraduate students, *International Journal of Computer Science & Engineering Survey*, vol.8, no.1, pp.1-18, 2017.

[4] A. Barrau and S. Bonnabel, Invariant Kalman filtering, *Annual Review of Control, Robotics, and Autonomous Systems*, vol.1, no.1, pp.237-257, 2018.

[5] J. Yang, Y. Liu and Z. Li, Unconstrained continuous control set model predictive control based on Kalman filter for active power filter, *International Journal of Innovative Computing, Information and Control*, vol.17, no.5, pp.1705-1716, 2021.

[6] C. Urrea and R. Agramonte, Kalman filter: Historical overview and review of its use in robotics 60 years after its creation, *Journal of Sensors*, vol.2021, pp.1-21, 9674015, 2021.

[7] A. Aravkin, J. V. Burke, L. Ljung, A. Lozano and G. Pillonetto, Generalized Kalman smoothing: Modeling and algorithms, *Automatica*, vol.86, pp.63-86, 2017.

[8] R. Dehghannasiri, X. Qian and E. R. Doughert, A Bayesian robust Kalman smoothing framework for state-space models with uncertain noise statistics, *EURASIP Journal on Advances in Signal Processing*, vol.2018, no.1, DOI: 10.1186/s13634-018-0577-1, 2018.

[9] C. Grudzien and M. Bocquet, A fast, single-iteration ensemble Kalman smoother for sequential data assimilation, *Geoscientific Model Development Discussions*, DOI: 10.5194/gmd-2021-306, 2021.

[10] S. Sharma, R. Kulkarni, A. Vishnoi and R. Gannavarpu, Wrapped phase denoising using adaptive kalman smoother algorithm, *Journal of Modern Optics*, vol.69, no.15, pp.838-849, 2022.

[11] Y. S. Shmaliy, S. Zhao and C. K. Ahn, Unbiased finite impulse response filtering: An iterative alternative to Kalman filtering ignoring noise and initial conditions, *IEEE Control Systems Magazine*, vol.37, no.5, pp.70-89, 2017.

[12] S. Zhao, Y. S. Shmaliy and F. Liu, Fast Kalman-like optimal FIR filter for time-variant systems with improved robustness, *ISA Transactions*, vol.80, pp.160-168, 2018.

[13] S. H. You, C. K. Ahn, Y. S. Shmaliy and S. Zhao, Fusion Kalman and weighted UFIR state estimator with improved accuracy, *IEEE Trans. Industrial Electronics*, vol.67, no.12, pp.10713-10722, 2020.

[14] P. S. Kim, Diverse derivation methods and expressions of discrete-time finite memory structure filter, *Engineering Letters*, vol.29, no.2, pp.658-667, 2021.

[15] S. Zhao, J. Wang, Y. Shmaliy and F. Fei, Discrete time q-lag maximum likelihood FIR smoothing and iterative recursive algorithm, *IEEE Trans. Signal Processing*, vol.69, no.11, pp.6342-6354, 2021.

[16] P. S. Kim, Two-stage Bayesian finite memory structure smoother for discrete-time systems, *ICIC Express Letters*, vol.15, no.3, pp.209-217, 2021.

[17] J. Mendel, *Lessons in Estimation Theory for Signal Processing, Communications, and Control*, Prentice-Hall, Englewood Cliffs, NJ, 1995.

[18] M. Brenna, F. Foiadelli and D. Zaninelli, DC motor drives, *Electrical Railway Transportation Systems*, pp.359-422, DOI: 10.1002/9781119386827.ch8, 2018.