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# A NOTE ON INTUITIONISTIC FUZZY IMPLICATIVE UP-FILTERS

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Abstract. *In this paper, we introduce the concept of intuitionistic fuzzy implicative UP-filters of UP-algebras and investigate their properties. Moreover, we discuss the relationship between intuitionistic fuzzy implicative UP-filters and fuzzy implicative UP-filters. Also, we establish the concept of the complement and the level subset of an intuitionistic fuzzy implicative UP-filter.*
**Keywords:** UP-algebra, Implicative UP-filter, UP-filter, Intuitionistic fuzzy implicative UP-filter, Intuitionistic fuzzy UP-filter

1. **Introduction.** Algebra structure has importance in mathematics with a wide range of research such as BCK-algebras [1], BCI-algebras [2], BCH-algebras [3], KU-algebras [4], and UP-algebras [5]. They are strongly connected with logic. In 2017, Iampan [5] introduced the concept of UP-algebras as a generalization of KU-algebras. Another interesting idea is UP-algebras, and many researchers brought this concept of UP-algebras into various concepts, such as UP-algebras with fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets, picture fuzzy sets, bipolar fuzzy sets, and neutrosophic sets. The expansion of the UP-algebra concept to a new notion has been attractive; for example, Somjanta

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et al. [6] introduced the notion of a UP-filter and discussed the fuzzy set theory of a UP-subalgebra, a UP-ideal and a UP-filter. In 2019, Jun and Iampan [7] introduced the concept of comparative and allied UP-filters and investigated several properties. They discussed a comparative UP-filter to be an implicative UP-filter. The concept of fuzzy sets by Zadeh [8] in 1965 displayed uncertainties that use mathematical tools and the importance of using a wide range of existing theories. Next, the concept of intuitionistic fuzzy sets as a generalization of the concept of fuzzy sets was proposed by Atanassov [9] in 1986. In 2015, Kesorn et al. [10] first introduced and studied the concept of intuitionistic fuzzy sets in UP-algebras. Later in 2019, Thongngam and Iampan [11] studied the concepts of intuitionistic fuzzy UP-filters and intuitionistic fuzzy near UP-filters in UP-algebras. In 2020, Abdullah and Shadhan [12] applied the concept of intuitionistic fuzzy sets on Q-algebras. In addition, Songsaeng et al. [13] have also studied neutrosophic implicative UP-filters of UP-algebras in 2021.

We are interested in extending the notion of implicative UP-filters to intuitionistic fuzzy implicative UP-filters to supplement the intuitionistic fuzzy set notion of UP-algebras. This article aims to introduce the new concept of intuitionistic fuzzy implicative UP-filters (IFIUPFs) detailed below and gives some definitions, properties, and examples of UP-algebras. As a result, we find a relationship between IFIUPFs and their level subsets and complements. Finally, we conclude and plan to future work.

2. **Preliminaries.** In this section, we present the concept of UP-algebras and other definitions used in the study of this article.

**Definition 2.1.** [5] *An algebra* $\tilde{M} = \left(\tilde{M}, \star, 0\right)$ *of type* $(2,0)$ *is called a UP-algebra, where* $\tilde{M}$ *is a nonempty set,* $\star$ *is a binary operation on* $\tilde{M}$*, and* $0$ *is a fixed element of* $\tilde{M}$ *if it satisfies the following axioms:*

$$\left(\textit{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left((\tilde{q} \star \tilde{r}) \star ((\tilde{p} \star \tilde{q}) \star (\tilde{p} \star \tilde{r})) = 0\right), \tag{1}$$

$$\left(\textit{for all } \tilde{p} \in \tilde{M}\right) \left(0 \star \tilde{p} = \tilde{p}\right), \tag{2}$$

$$\left(\textit{for all } \tilde{p} \in \tilde{M}\right) \left(\tilde{p} \star 0 = 0\right), \tag{3}$$

$$\left(\textit{for all } \tilde{p}, \tilde{q} \in \tilde{M}\right) \left(\tilde{p} \star \tilde{q} = 0, \tilde{q} \star \tilde{p} = 0 \Rightarrow \tilde{p} = \tilde{q}\right). \tag{4}$$

From [5], we know that the concept of UP-algebras is a generalization of KU-algebras. For ease of study, we write $\tilde{M}$ instead of a UP-algebra $\left(\tilde{M}, \star, 0\right)$.

The binary relation $\leq$ on $\tilde{M}$ is defined as follows: $\left(\text{for all } \tilde{p}, \tilde{q} \in \tilde{M}\right) \left(\tilde{p} \leq \tilde{q} \Leftrightarrow \tilde{p} \star \tilde{q} = 0\right)$ and the following assertions are valid (see [5, 14]).

$$\left(\text{for all } \tilde{p} \in \tilde{M}\right) \left(\tilde{p} \leq \tilde{p}\right), \tag{5}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\tilde{p} \leq \tilde{q}, \tilde{q} \leq \tilde{r} \Rightarrow \tilde{p} \leq \tilde{r}\right), \tag{6}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\tilde{p} \leq \tilde{q} \Rightarrow \tilde{r} \star \tilde{p} \leq \tilde{r} \star \tilde{q}\right), \tag{7}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\tilde{p} \leq \tilde{q} \Rightarrow \tilde{q} \star \tilde{r} \leq \tilde{p} \star \tilde{r}\right), \tag{8}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\tilde{p} \leq \tilde{q} \star \tilde{p}, \text{ in particular, } \tilde{q} \star \tilde{r} \leq \tilde{p} \star (\tilde{q} \star \tilde{r})\right), \tag{9}$$

$$\left(\text{for all } \tilde{p}, \tilde{q} \in \tilde{M}\right) \left(\tilde{q} \star \tilde{p} \leq \tilde{p} \Leftrightarrow \tilde{p} = \tilde{q} \star \tilde{p}\right), \tag{10}$$

$$\left(\text{for all } \tilde{p}, \tilde{q} \in \tilde{M}\right) \left(\tilde{p} \leq \tilde{q} \star \tilde{q}\right), \tag{11}$$

$$\left(\text{for all } \tilde{a}, \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\tilde{p} \star (\tilde{q} \star \tilde{r}) \leq \tilde{p} \star ((\tilde{a} \star \tilde{q}) \star (\tilde{a} \star \tilde{r}))\right), \tag{12}$$

$$\left(\text{for all } \tilde{a}, \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(((\tilde{a} \star \tilde{p}) \star (\tilde{a} \star \tilde{q})) \star \tilde{r} \leq (\tilde{p} \star \tilde{q}) \star \tilde{r}\right), \tag{13}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left((\tilde{p} \star \tilde{q}) \star \tilde{r} \leq \tilde{q} \star \tilde{r}\right), \tag{14}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\tilde{p} \leq \tilde{q} \Rightarrow \tilde{p} \leq \tilde{r} \star \tilde{q}\right), \tag{15}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left((\tilde{p} \star \tilde{q}) \star \tilde{r} \leq \tilde{p} \star (\tilde{q} \star \tilde{r})\right), \tag{16}$$

$$\left(\text{for all } \tilde{a}, \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left((\tilde{p} \star \tilde{q}) \star \tilde{r} \leq \tilde{q} \star (\tilde{a} \star \tilde{r})\right). \tag{17}$$

You may find further UP-algebra studies and examples at [15, 16, 17, 18, 19, 20].

**Definition 2.2.** *A nonempty subset $\tilde{S}$ of $\tilde{M}$ is called*

*(1) a UP-subalgebra (UPS) of $\tilde{M}$ if*

$$\left(\text{for all } \tilde{p}, \tilde{q} \in \tilde{S}\right) \left(\tilde{p} \star \tilde{q} \in \tilde{S}\right), \tag{18}$$

*(2) a UP-ideal (UPI) of $\tilde{M}$ if*

$$0 \in \tilde{S}, \tag{19}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\tilde{p} \star (\tilde{q} \star \tilde{r}) \in \tilde{S}, \tilde{q} \in \tilde{S} \Rightarrow \tilde{p} \star \tilde{r} \in \tilde{S}\right), \tag{20}$$

*(3) a UP-filter (UPF) of $\tilde{M}$ if* (19) *and*

$$\left(\text{for all } \tilde{p}, \tilde{q} \in \tilde{M}\right) \left(\tilde{p} \in \tilde{S}, \tilde{p} \star \tilde{q} \in \tilde{S} \Rightarrow \tilde{q} \in \tilde{S}\right), \tag{21}$$

*(4) an implicative UP-filter (IUPF) of $\tilde{M}$ if* (19) *and*

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\tilde{p} \star (\tilde{q} \star \tilde{r}) \in \tilde{S}, \tilde{p} \star \tilde{q} \in \tilde{S} \Rightarrow \tilde{p} \star \tilde{r} \in \tilde{S}\right). \tag{22}$$

From [21], Jun and Iampan show that every IUPF is a UPF, but the converse is not true in general.

**Theorem 2.1.** [17] *Let $\mathfrak{K}$ be a nonempty family of UPFs (resp., IUPFs) of $\tilde{M}$. Then $\cap\mathfrak{K}$ is a UPF (resp., IUPF) of $\tilde{M}$.*

Now, we review the concepts of fuzzy sets and intuitionistic fuzzy sets.

A *fuzzy set* (FS) $\omega$ in a nonempty set $\tilde{S}$ is a function from $\tilde{S}$ into the unit closed interval $[0, 1]$ of real numbers, i.e., $\omega : \tilde{S} \to [0, 1]$. For any two FSs $\omega_1$ and $\omega_2$ in $\tilde{S}$, we define (1) $\omega_1 \geq \omega_2 \Leftrightarrow \omega_1(\tilde{p}) \geq \omega_2(\tilde{p})$ for all $\tilde{p} \in \tilde{S}$, (2) $\omega_1 = \omega_2 \Leftrightarrow \omega_1 \geq \omega_2$ and $\omega_2 \geq \omega_1$, (3) $(\omega_1 \wedge \omega_2)(\tilde{p}) = \min\{\omega_1(\tilde{p}), \omega_2(\tilde{p})\}$ for all $\tilde{p} \in \tilde{S}$. Let $\omega$ be an FS in $\tilde{S}$. The FS $\overline{\omega}$ is defined by $\overline{\omega}(\tilde{p}) = 1 - \omega(\tilde{p})$ for all $\tilde{p} \in \tilde{S}$. We call $\overline{\omega}$ as the *complement* of $\omega$ in $\tilde{S}$.

**Definition 2.3.** *An FS $\omega$ in $\tilde{M}$ is called*

*(1) a fuzzy UP-subalgebra (FUPS) of $\tilde{M}$ if*

$$\left(\text{for all } \tilde{p}, \tilde{q} \in \tilde{M}\right) \left(\omega(\tilde{p} \star \tilde{q}) \geq \min\{\omega(\tilde{p}), \omega(\tilde{q})\}\right), \tag{23}$$

*(2) a fuzzy UP-ideal (FUPI) of $\tilde{M}$ if*

$$\left(\text{for all } \tilde{p} \in \tilde{M}\right) \left(\omega(0) \geq \omega(\tilde{p})\right), \tag{24}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\omega(\tilde{p} \star \tilde{r}) \geq \min\{\omega(\tilde{p} \star (\tilde{q} \star \tilde{r})), \omega(\tilde{q})\}\right), \tag{25}$$

*(3) a fuzzy UP-filter (FUPF) of $\tilde{M}$ if* (24) *and*

$$\left(\text{for all } \tilde{p}, \tilde{q} \in \tilde{M}\right) \left(\omega(\tilde{q}) \geq \min\left\{\omega(\tilde{p}), \omega(\tilde{p} \star \tilde{q})\right\}\right), \tag{26}$$

*(4) a fuzzy implicative UP-filter (FIUPF) of $\tilde{M}$ if* (24) *and*

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\omega(\tilde{p} \star \tilde{r}) \geq \min\left\{\omega\left(\tilde{p} \star (\tilde{q} \star \tilde{r})\right), \omega(\tilde{p} \star \tilde{q})\right\}\right). \tag{27}$$

An *intuitionistic fuzzy set* (IFS) in a nonempty set $\tilde{S}$ is an object having the form $F = \left\{(\tilde{p}, \omega_F(\tilde{p}), \delta_F(\tilde{p})) \,\middle|\, \tilde{p} \in \tilde{S}\right\}$, where $\omega_F : \tilde{S} \to [0,1]$ and $\delta_F : \tilde{S} \to [0,1]$ denote the degree of membership and degree of nonmembership, respectively, and $\left(\text{for all } \tilde{p} \in \tilde{S}\right) (0 \leq \omega_F(\tilde{p}) + \delta_F(\tilde{p}) \leq 1)$. We shall use the symbol $F = (\omega_F, \delta_F)$ for the IFS $F = \left\{(\tilde{p}, \omega_F(\tilde{p}), \delta_F(\tilde{p})) \,\middle|\, \tilde{p} \in \tilde{S}\right\}$ for the sake of notational simplicity.

Kesorn et al. [10] and Thongngam and Iampan [11] introduced the concepts of intuitionistic fuzzy UP-subalgebras, intuitionistic fuzzy UP-ideals, and intuitionistic fuzzy UP-filters of UP-algebras as follows.

**Definition 2.4.** [10] *An IFS $F = (\omega_F, \delta_F)$ in $\tilde{M}$ is called an intuitionistic fuzzy UP-subalgebra (IFUPS) of $\tilde{M}$ if*

$$\left(\text{for all } \tilde{p}, \tilde{q} \in \tilde{M}\right) \left(\omega_F(\tilde{p} \star \tilde{q}) \geq \min\left\{\omega_F(\tilde{p}), \omega_F(\tilde{q})\right\}\right), \tag{28}$$

$$\left(\text{for all } \tilde{p}, \tilde{q} \in \tilde{M}\right) \left(\delta_F(\tilde{p} \star \tilde{q}) \leq \max\left\{\delta_F(\tilde{p}), \delta_F(\tilde{q})\right\}\right). \tag{29}$$

**Definition 2.5.** [10] *An IFS $F = (\omega_F, \delta_F)$ in $\tilde{M}$ is called an intuitionistic fuzzy UP-ideal (IFUPI) of $\tilde{M}$ if*

$$\left(\text{for all } \tilde{p} \in \tilde{M}\right) \left(\omega_F(0) \geq \omega_F(\tilde{p})\right), \tag{30}$$

$$\left(\text{for all } \tilde{p} \in \tilde{M}\right) \left(\delta_F(0) \leq \delta_F(\tilde{p})\right), \tag{31}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\omega_F(\tilde{p} \star \tilde{r}) \geq \min\left\{\omega_F\left(\tilde{p} \star (\tilde{q} \star \tilde{r})\right), \omega_F(\tilde{q})\right\}\right), \tag{32}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\delta_F(\tilde{p} \star \tilde{r}) \leq \max\left\{\delta_F\left(\tilde{p} \star (\tilde{q} \star \tilde{r})\right), \delta_F(\tilde{q})\right\}\right). \tag{33}$$

**Definition 2.6.** [11] *An IFS $F = (\omega_F, \delta_F)$ in $\tilde{M}$ is called an intuitionistic fuzzy UP-filter (IFUPF) of $\tilde{M}$ if* (30), (31), *and*

$$\left(\text{for all } \tilde{p}, \tilde{q} \in \tilde{M}\right) \left(\omega_F(\tilde{q}) \geq \min\left\{\omega_F(\tilde{p} \star \tilde{q}), \omega_F(\tilde{p})\right\}\right), \tag{34}$$

$$\left(\text{for all } \tilde{p}, \tilde{q} \in \tilde{M}\right) \left(\delta_F(\tilde{q}) \leq \max\left\{\delta_F(\tilde{p} \star \tilde{q}), \delta_F(\tilde{p})\right\}\right). \tag{35}$$

3. **Intuitionistic Fuzzy Implicative UP-Filters.** In this section, we introduce the concept of IFIUPFs, and we investigate properties of IFIUPFs in UP-algebras.

**Definition 3.1.** *An IFS $F = (\omega_F, \delta_F)$ in $\tilde{M}$ is called an intuitionistic fuzzy implicative UP-filter (IFIUPF) of $\tilde{M}$ if* (30), (31), *and*

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\omega_F(\tilde{p} \star \tilde{r}) \geq \min\left\{\omega_F\left(\tilde{p} \star (\tilde{q} \star \tilde{r})\right), \omega_F(\tilde{p} \star \tilde{q})\right\}\right), \tag{36}$$

$$\left(\text{for all } \tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}\right) \left(\delta_F(\tilde{p} \star \tilde{r}) \leq \max\left\{\delta_F\left(\tilde{p} \star (\tilde{q} \star \tilde{r})\right), \delta_F(\tilde{p} \star \tilde{q})\right\}\right). \tag{37}$$

**Example 3.1.** *Consider a UP-algebra* $\tilde{M} = \{0, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4\}$ *with the following Cayley table:*

| $\star$ | $0$ | $\tilde{q}_1$ | $\tilde{q}_2$ | $\tilde{q}_3$ | $\tilde{q}_4$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $\tilde{q}_1$ | $\tilde{q}_2$ | $\tilde{q}_3$ | $\tilde{q}_4$ |
| $\tilde{q}_1$ | $0$ | $0$ | $0$ | $\tilde{q}_1$ | $\tilde{q}_2$ |
| $\tilde{q}_2$ | $0$ | $\tilde{q}_1$ | $0$ | $\tilde{q}_4$ | $\tilde{q}_1$ |
| $\tilde{q}_3$ | $0$ | $\tilde{q}_2$ | $\tilde{q}_4$ | $0$ | $0$ |
| $\tilde{q}_4$ | $0$ | $0$ | $0$ | $\tilde{q}_1$ | $0$ |

*Define an IFS* $F = (\omega_F, \delta_F)$ *in* $\tilde{M}$ *as follows:*

| $A$ | $0$ | $\tilde{q}_1$ | $\tilde{q}_2$ | $\tilde{q}_3$ | $\tilde{q}_4$ |
|---|---|---|---|---|---|
| $\omega_F$ | 0.2 | 0.4 | 0.2 | 0.5 | 0.7 |
| $\delta_F$ | 0.1 | 0.2 | 0.1 | 0.4 | 0.5 |

*Then* $F = (\omega_F, \delta_F)$ *is an IFIUPF of* $\tilde{M}$*.*

**Theorem 3.1.** *Every IFIUPF of* $\tilde{M}$ *is an IFUPF.*

**Proof:** Let $F = (\omega_F, \delta_F)$ be an IFIUPF of $\tilde{M}$. Then for all $\tilde{p}, \tilde{q} \in \tilde{M}$, $\omega_F(0) \geq \omega_F(\tilde{p})$, $\delta_F(0) \leq \delta_F(\tilde{p})$, $\omega_F(\tilde{q}) = \omega_F(0 \star \tilde{q}) \geq \min\{\omega_F(0 \star (\tilde{p} \star \tilde{q})), \omega_F(0 \star \tilde{p})\} = \min\{\omega_F(\tilde{p} \star \tilde{q}), \omega_F(\tilde{p})\}$, and $\delta_F(\tilde{q}) = \delta_F(0 \star \tilde{q}) \leq \max\{\delta_F(0 \star (\tilde{p} \star \tilde{q})), \delta_F(0 \star \tilde{p})\} = \max\{\delta_F(\tilde{p} \star \tilde{q}), \delta_F(\tilde{p})\}$. Hence, $F = (\omega_F, \delta_F)$ is an IFUPF of $\tilde{M}$. □

**Example 3.2.** *Consider a UP-algebra* $\tilde{M} = \{0, \tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4\}$ *with the following Cayley table:*

| $\star$ | $0$ | $\tilde{r}_1$ | $\tilde{r}_2$ | $\tilde{r}_3$ | $\tilde{r}_4$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $\tilde{r}_1$ | $\tilde{r}_2$ | $\tilde{r}_3$ | $\tilde{r}_4$ |
| $\tilde{r}_1$ | $0$ | $0$ | $0$ | $\tilde{r}_1$ | $\tilde{r}_4$ |
| $\tilde{r}_2$ | $0$ | $\tilde{r}_1$ | $0$ | $\tilde{r}_4$ | $\tilde{r}_1$ |
| $\tilde{r}_3$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\tilde{r}_4$ | $0$ | $0$ | $0$ | $\tilde{r}_1$ | $0$ |

*Define an IFS* $F = (\omega_F, \delta_F)$ *in* $\tilde{M}$ *as follows:*

| $\tilde{M}$ | $0$ | $\tilde{r}_1$ | $\tilde{r}_2$ | $\tilde{r}_3$ | $\tilde{r}_4$ |
|---|---|---|---|---|---|
| $\omega_F$ | 0.4 | 0.4 | 0.1 | 0.3 | 0.1 |
| $\delta_F$ | 0.3 | 0.3 | 0.2 | 0.3 | 0.2 |

*Then* $F = (\omega_F, \delta_F)$ *is an IFUPF of* $\tilde{M}$*, but it is not an IFIUPF of* $\tilde{M}$*. Indeed,* $\omega_F(0 \star \tilde{r}_3) = 0.3 < 0.4 = \min\{\omega_F(0 \star (\tilde{r}_1 \star \tilde{r}_3)), \omega_F(0 \star \tilde{r}_1)\}$.

**Theorem 3.2.** *If an IFS* $F = (\omega_F, \delta_F)$ *in* $\tilde{M}$ *is constant, then* $F = (\omega_F, \delta_F)$ *is an IFIUPF of* $\tilde{M}$*.*

**Proof:** Suppose that $F = (\omega_F, \delta_F)$ is a constant IFS in $\tilde{M}$. Then there exist elements $m$, $n$ in $[0, 1]$ such that $\omega_F(\tilde{p}) = m$ and $\delta_F(\tilde{p}) = n$ for all $\tilde{p} \in \tilde{M}$. Thus, $\omega_F(0) = m = \omega_F(\tilde{p})$ and $\delta_F(0) = n = \delta_F(\tilde{p})$ for all $\tilde{p} \in \tilde{M}$. For all $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}$, $\omega_F(\tilde{p} \star \tilde{r}) = m = \min\{m, m\} = \min\{\omega_F(\tilde{p} \star (\tilde{q} \star \tilde{r})), \omega_F(\tilde{p} \star \tilde{q})\}$ and $\delta_F(\tilde{p} \star \tilde{r}) = n = \max\{n, n\} = \max\{\delta_F(\tilde{p} \star (\tilde{q} \star \tilde{r})), \delta_F(\tilde{p} \star \tilde{q})\}$. Hence, $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$. □

**Theorem 3.3.** *An IFS* $F = (\omega_F, \delta_F)$ *is an IFIUPF of* $\tilde{M}$ *if and only if the FSs* $\omega_F$ *and* $\overline{\delta_F}$ *are FIUPFs of* $\tilde{M}$*.*

**Proof:** Suppose that an IFS $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$. Then $\omega_F(0) \geq \omega_F(\tilde{p})$ and $\omega_F(\tilde{p} \star \tilde{r}) \geq \min\{\omega_F(\tilde{p} \star (\tilde{q} \star \tilde{r})), \omega_F(\tilde{p} \star \tilde{q})\}$ for all $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}$. Thus, $\omega_F$ is an FIUPF of $\tilde{M}$. Let $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}$. Since $\delta_F(0) \leq \delta_F(\tilde{p})$ and $\delta_F(\tilde{p} \star \tilde{r}) \leq \max\{\delta_F(\tilde{p} \star (\tilde{q} \star \tilde{r})), \delta_F(\tilde{p} \star \tilde{q})\}$, we have $\overline{\delta_F}(0) = 1 - \delta_F(0) \geq 1 - \delta_F(\tilde{p}) = \overline{\delta_F}(\tilde{p})$ and $\overline{\delta_F}(\tilde{p} \star \tilde{r}) = 1 - \delta_F(\tilde{p} \star \tilde{r}) \geq$

$1 - \max\{\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \delta_F(\tilde{p}\star\tilde{q})\} = \min\{1-\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), 1-\delta_F(\tilde{p}\star\tilde{q})\} = \min\{\overline{\delta_F}(\tilde{p}\star(\tilde{q}\star\tilde{r})), \overline{\delta_F}(\tilde{p}\star\tilde{q})\}$. Thus, $\overline{\delta_F}$ is an FIUPF of $\tilde{M}$.

Conversely, suppose that the FSs $\omega_F$ and $\overline{\delta_F}$ are FIUPFs of $\tilde{M}$. Then $\omega_F(0) \geq \omega_F(\tilde{p})$ and $\omega_F(\tilde{p}\star\tilde{r}) \geq \min\{\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \omega_F(\tilde{p}\star\tilde{q})\}$ for all $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$. Let $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$. Since $\overline{\delta_F}(0) \geq \overline{\delta_F}(\tilde{p})$ and $\overline{\delta_F}(\tilde{p}\star\tilde{r}) \geq \min\{\overline{\delta_F}(\tilde{p}\star(\tilde{q}\star\tilde{r})), \overline{\delta_F}(\tilde{p}\star\tilde{q})\}$, we have $1-\delta_F(0) \geq 1-\delta_F(\tilde{p})$. Thus, $\delta_F(0) \leq \delta_F(\tilde{p})$ and $1-\delta_F(\tilde{p}\star\tilde{r}) \geq \min\{1-\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), 1-\delta_F(\tilde{p}\star\tilde{q})\} = 1-\max\{\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \delta_F(\tilde{p}\star\tilde{q})\}$, so $\delta_F(\tilde{p}\star\tilde{r}) \leq \max\{\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \delta_F(\tilde{p}\star\tilde{q})\}$. Hence, $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$. □

**Theorem 3.4.** *An IFS $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$ if and only if the IFSs $\Box F = (\omega_F, \overline{\omega_F})$ and $\Diamond F = (\overline{\delta_F}, \delta_F)$ are IFIUPFs of $\tilde{M}$.*

**Proof:** Suppose that an IFS $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$, then $\omega_F(0) \geq \omega_F(\tilde{p})$ and $\omega_F(\tilde{p}\star\tilde{r}) \geq \min\{\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \omega_F(\tilde{p}\star\tilde{q})\}$ for all $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$. Thus, for all $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$, $\overline{\omega_F}(0) = 1-\omega_F(0) \leq 1-\omega_F(\tilde{p}) = \overline{\omega_F}(\tilde{p})$ and $\overline{\omega_F}(\tilde{p}\star\tilde{r}) = 1-\omega_F(\tilde{p}\star\tilde{r}) \leq 1-\min\{\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \omega_F(\tilde{p}\star\tilde{q})\} = \max\{1-\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), 1-\omega_F(\tilde{p}\star\tilde{q})\} = \max\{\overline{\omega_F}(\tilde{p}\star(\tilde{q}\star\tilde{r})), \overline{\omega_F}(\tilde{p}\star\tilde{q})\}$. Hence, $\Box F = (\omega_F, \overline{\omega_F})$ is an IFIUPF of $\tilde{M}$. Let $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$. Then $\delta_F(0) \leq \delta_F(\tilde{p})$ and $\delta_F(\tilde{p}\star\tilde{r}) \leq \max\{\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \delta_F(\tilde{p}\star\tilde{q})\}$. Thus, for all $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$, $\overline{\delta_F}(0) = 1-\delta_F(0) \geq 1-\delta_F(\tilde{p}) = \overline{\delta_F}(\tilde{p})$ and $\overline{\delta_F}(\tilde{p}\star\tilde{r}) = 1-\delta_F(\tilde{p}\star\tilde{r}) \geq 1-\max\{\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \delta_F(\tilde{p}\star\tilde{q})\} = \min\{1-\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), 1-\delta_F(\tilde{p}\star\tilde{q})\} = \min\{\overline{\delta_F}(\tilde{p}\star(\tilde{q}\star\tilde{r})), \overline{\delta_F}(\tilde{p}\star\tilde{q})\}$. Hence, $\Diamond F = (\overline{\delta_F}, \delta_F)$ is an IFIUPF of $\tilde{M}$.

Conversely, suppose that the IFSs $\Box F = (\omega_F, \overline{\omega_F})$ and $\Diamond F = (\overline{\delta_F}, \delta_F)$ are IFIUPFs of $\tilde{M}$. Then for any $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$, $\omega_F(0) \geq \omega_F(\tilde{p})$, $\omega_F(\tilde{p}\star\tilde{r}) \geq \min\{\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \omega_F(\tilde{p}\star\tilde{q})\}$, $\delta_F(0) \leq \delta_F(\tilde{p})$, and $\delta_F(\tilde{p}\star\tilde{r}) \leq \max\{\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \delta_F(\tilde{p}\star\tilde{q})\}$. Hence, $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$. □

**Theorem 3.5.** *An IFS $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$ if and only if the IFS $\triangle F = (\overline{\delta_F}, \overline{\omega_F})$ is an IFIUPF of $\tilde{M}$.*

**Proof:** Suppose that an IFS $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$. Then for any $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$, $\omega_F(0) \leq \omega_F(\tilde{p})$ and $\omega_F(\tilde{p}\star\tilde{r}) \geq \min\{\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \omega_F(\tilde{p}\star\tilde{q})\}$. Thus, for any $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$, $\overline{\omega_F}(0) = 1-\omega_F(0) \geq 1-\omega_F(\tilde{p}) = \overline{\omega_F}(\tilde{p})$ and $\overline{\omega_F}(\tilde{p}\star\tilde{r}) = 1-\omega_F(\tilde{p}\star\tilde{r}) \leq 1-\min\{\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \omega_F(\tilde{p}\star\tilde{q})\} = \max\{1-\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), 1-\omega_F(\tilde{p}\star\tilde{q})\} = \max\{\overline{\omega_F}(\tilde{p}\star(\tilde{q}\star\tilde{r})), \overline{\omega_F}(\tilde{p}\star\tilde{q})\}$. Now, for any $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$, $\delta_F(0) \leq \delta_F(\tilde{p})$ and $\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})) \leq \max\{\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \delta_F(\tilde{p}\star\tilde{q})\}$. Thus, for any $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$, $\overline{\delta_F}(0) = 1-\delta_F(0) \geq 1-\delta_F(\tilde{p}) = \overline{\delta_F}(\tilde{p})$ and $\overline{\delta_F}(\tilde{p}\star\tilde{r}) = 1-\delta_F(\tilde{p}\star\tilde{r}) \geq 1-\max\{\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \delta_F(\tilde{p}\star\tilde{q})\} = \min\{1-\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), 1-\delta_F(\tilde{p}\star\tilde{q})\} = \min\{\overline{\delta_F}(\tilde{p}\star(\tilde{q}\star\tilde{r})), \overline{\delta_F}(\tilde{p}\star\tilde{q})\}$. Hence, $\triangle F = (\overline{\delta_F}, \overline{\omega_F})$ is an IFIUPF of $\tilde{M}$.

Conversely, suppose that the $\triangle F = (\overline{\delta_F}, \overline{\omega_F})$ is an IFIUPF of $\tilde{M}$. Then for any $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$, $\overline{\delta_F}(0) \geq \overline{\delta_F}(\tilde{p})$ and $\overline{\delta_F}(\tilde{p}\star\tilde{r}) \geq \min\{\overline{\delta_F}(\tilde{p}\star(\tilde{q}\star\tilde{r})), \overline{\delta_F}(\tilde{p}\star\tilde{q})\}$. Thus, for any $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$, $1-\delta_F(0) \geq 1-\delta_F(\tilde{p})$ and $1-\delta_F(\tilde{p}\star\tilde{r}) \geq 1-\max\{\delta(\tilde{p}\star(\tilde{q}\star\tilde{r})), \delta(\tilde{p}\star\tilde{q})\}$, so $\delta_F(0) \leq \delta_F(\tilde{p})$ and $\delta_F(\tilde{p}\star\tilde{r}) \leq \max\{\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \delta_F(\tilde{p}\star\tilde{q})\}$. Now, for any $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$, $\overline{\omega_F}(0) \leq \overline{\omega_F}(\tilde{p})$ and $\overline{\omega_F}(\tilde{p}\star\tilde{r}) \leq \max\{\overline{\omega_F}(\tilde{p}\star(\tilde{q}\star\tilde{r})), \overline{\omega_F}(\tilde{p}\star\tilde{q})\}$. Thus, for any $\tilde{p},\tilde{q},\tilde{r} \in \tilde{M}$, $1-\omega_F(0) \geq 1-\omega_F(\tilde{p})$ and $1-\omega_F(\tilde{p}\star\tilde{r}) \leq 1-\min\{\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \omega_F(\tilde{p}\star\tilde{q})\}$, so $\omega_F(0) \geq \omega_F(\tilde{p})$ and $\omega_F(\tilde{p}\star\tilde{r}) \geq \min\{\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \omega_F(\tilde{p}\star\tilde{q})\}$. Hence, $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$. □

For a nonempty subset $\tilde{S}$ of a nonempty set $\tilde{M}$, the characteristic function $f_{\tilde{S}}$ of $\tilde{M}$ is a function of $\tilde{M}$ into $\{0,1\}$ defined as follows: for all $\tilde{p} \in \tilde{M}$, $f_{\tilde{S}}(\tilde{p}) = \begin{cases} 1 & \text{if } \tilde{p} \in \tilde{S} \\ 0 & \text{if } \tilde{p} \notin \tilde{S} \end{cases}$.

Then for all $\tilde{p} \in \tilde{M}$, $\overline{f_{\tilde{S}}}(\tilde{p}) = \begin{cases} 1 & \text{if } \tilde{p} \notin \tilde{S} \\ 0 & \text{if } \tilde{p} \in \tilde{S} \end{cases}$. We denote the IFS in $\tilde{M}$ with the degree of membership $f_{\tilde{S}}$ and the degree of nonmembership $\overline{f_{\tilde{S}}}$ by $F_{\tilde{S}}$, that is, $F_{\tilde{S}} = \left(f_{\tilde{S}}, \overline{f_{\tilde{S}}}\right)$.

**Theorem 3.6.** *A nonempty subset $\tilde{S}$ of $\tilde{M}$ is an IUPF of $\tilde{M}$ if and only if the IFS $F_{\tilde{S}} = \left(f_{\tilde{S}}, \overline{f_{\tilde{S}}}\right)$ is an IFIUPF of $\tilde{M}$.*

**Proof:** Assume that $\tilde{S}$ is an IUPF of $\tilde{M}$. Since $0 \in \tilde{S}$, we have $f_{\tilde{S}}(0) = 1 \geq f_{\tilde{S}}(\tilde{p})$ and $\overline{f_{\tilde{S}}}(0) = 0 \leq \overline{f_{\tilde{S}}}(\tilde{p})$ for all $\tilde{p} \in \tilde{M}$. Let $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{S}$. In the case that $\tilde{p} \star (\tilde{q} \star \tilde{r}) \notin \tilde{S}$ or $\tilde{p} \star \tilde{q} \notin \tilde{S}$, we have $f_{\tilde{S}}(\tilde{p} \star \tilde{r}) \geq 0 = \min\{f_{\tilde{S}}(\tilde{p} \star (\tilde{q} \star \tilde{r})), f_{\tilde{S}}(\tilde{p} \star \tilde{q})\}$ and $\overline{f_{\tilde{S}}}(\tilde{p} \star \tilde{r}) \leq 1 = \max\left\{\overline{f_{\tilde{S}}}(\tilde{p} \star (\tilde{q} \star \tilde{r})), \overline{f_{\tilde{S}}}(\tilde{p} \star \tilde{q})\right\}$. On the other hand, $\tilde{p} \star (\tilde{q} \star \tilde{r}) \in \tilde{S}$ and $\tilde{p} \star \tilde{q} \in \tilde{S}$. By the assumption, we have $\tilde{p} \star \tilde{r} \in \tilde{S}$. Hence, $f_{\tilde{S}}(\tilde{p} \star \tilde{r}) = 1 = \min\{f_{\tilde{S}}(\tilde{p} \star (\tilde{q} \star \tilde{r})), f_{\tilde{S}}(\tilde{p} \star \tilde{q})\}$ and $\overline{f_{\tilde{S}}}(\tilde{p} \star \tilde{r}) = 0 = \max\left\{\overline{f_{\tilde{S}}}(\tilde{p} \star (\tilde{q} \star \tilde{r})), \overline{f_{\tilde{S}}}(\tilde{p} \star \tilde{q})\right\}$. Therefore, $F_{\tilde{S}} = \left(f_{\tilde{S}}, \overline{f_{\tilde{S}}}\right)$ is an IFIUPF of $\tilde{M}$.

Conversely, $F_{\tilde{S}} = \left(f_{\tilde{S}}, \overline{f_{\tilde{S}}}\right)$ is an IFIUPF of $\tilde{M}$. Then $f_{\tilde{S}}(0) \geq f_{\tilde{S}}(\tilde{p}) = 1$ when $\tilde{p} \in \tilde{S}$. Thus, $0 \in \tilde{S}$. Let $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{S}$ be such that $\tilde{p} \star (\tilde{q} \star \tilde{r}) \in \tilde{S}$ and $\tilde{p} \star \tilde{q} \in \tilde{S}$. By the assumption, we have $f_{\tilde{S}}(\tilde{p} \star \tilde{r}) = \min\{f_{\tilde{S}}(\tilde{p} \star (\tilde{q} \star \tilde{r})), f_{\tilde{S}}(\tilde{p} \star \tilde{q})\} = 1$. Hence, $\tilde{p} \star \tilde{r} \in \tilde{S}$. Therefore, $\tilde{S}$ is an IUPF of $\tilde{M}$. □

**Definition 3.2.** *Let $\omega$ and $\delta$ be FSs in a nonempty set $\tilde{M}$. For $s, t \in [0,1]$, the set $\ddot{\mathfrak{U}}(\omega; t) = \left\{\tilde{p} \in \tilde{M} \middle| \omega(\tilde{p}) \geq t\right\}$ and $\ddot{\mathfrak{U}}^{+}(\omega; t) = \left\{\tilde{p} \in \tilde{M} \middle| \omega(\tilde{p}) > t\right\}$ are called an* upper $t$-level subset *and an* upper $t$-strong level subset *of $\omega$, respectively. The set $\ddot{\mathfrak{L}}(\omega; s) = \left\{\tilde{p} \in \tilde{M} \middle| \omega(\tilde{p}) \leq s\right\}$ and $\ddot{\mathfrak{L}}^{-}(\omega; s) = \left\{\tilde{p} \in \tilde{M} \middle| \omega(\tilde{p}) < s\right\}$ are called a* lower $s$-level subset *and a* lower $s$-strong level subset *of $\omega$, respectively. The set $\ddot{\mathfrak{C}}(\omega, \delta; t, s) = \ddot{\mathfrak{U}}(\omega; t) \cap \ddot{\mathfrak{L}}(\delta; s)$ is called the* $(t,s)$-cut *of $\omega$ and $\delta$.*

**Theorem 3.7.** *An IFS $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M} = \left(\tilde{M}, \star, 0\right)$ if and only if the sets $\ddot{\mathfrak{U}}(\omega_F; t)$ and $\ddot{\mathfrak{L}}(\delta_F; s)$ are IUPFs of $\tilde{M}$ for each $s, t \in [0,1]$ such that $\ddot{\mathfrak{U}}(\omega_F; t) \neq \emptyset$ and $\ddot{\mathfrak{L}}(\delta_F; s) \neq \emptyset$.*

**Proof:** Suppose that an IFS $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$, let $s, t \in [0,1]$ be such that $\ddot{\mathfrak{U}}(\omega_F; t)$ and $\ddot{\mathfrak{L}}(\delta_F, s)$ are nonempty subset of $\tilde{M}$. Then there exist $\tilde{a} \in \ddot{\mathfrak{U}}(\omega_F; t)$ and $\tilde{b} \in \ddot{\mathfrak{L}}(\delta; s)$. Then $\omega(\tilde{a}) \geq t$ and $\delta_F\left(\tilde{b}\right) \leq s$. By assumption, we have $\omega_F(0) \geq \omega_F(\tilde{a}) \geq t$ and $\delta_F(0) \leq \delta_F\left(\tilde{b}\right) \leq s$, so $0 \in \ddot{\mathfrak{U}}(\omega_F; t)$ and $0 \in \ddot{\mathfrak{L}}(\delta_F; s)$. Let $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}$ be such that $\tilde{p} \star (\tilde{q} \star \tilde{r}) \in \ddot{\mathfrak{U}}(\omega; t)$ and $\tilde{p} \star \tilde{q} \in \ddot{\mathfrak{U}}(\omega; t)$. Then $\omega_F(\tilde{p} \star (\tilde{q} \star \tilde{r})) \geq t$ and $\omega(\tilde{p} \star \tilde{q}) \geq t$. Thus, $\omega_F(\tilde{p} \star \tilde{r}) \geq \min\{\omega_F(\tilde{p} \star (\tilde{q} \star \tilde{r})), \omega_F(\tilde{p} \star \tilde{q})\} \geq t$. So, $\tilde{p} \star \tilde{r} \in \ddot{\mathfrak{U}}(\omega_F; t)$. Hence, $\ddot{\mathfrak{U}}(\omega; t)$ is an IUPF of $\tilde{M}$. Finally, let $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}$ be such that $\tilde{p} \star (\tilde{q} \star \tilde{r}) \in \ddot{\mathfrak{L}}(\delta_F; s)$ and $\tilde{p} \star \tilde{q} \in \ddot{\mathfrak{L}}(\delta_F; s)$. Then $\delta_F(\tilde{p} \star (\tilde{q} \star \tilde{r})) \leq s$ and $\delta_F(\tilde{p} \star \tilde{q}) \leq s$. Thus, $\delta_F(\tilde{p} \star \tilde{r}) \leq \max\{\delta_F(\tilde{p} \star (\tilde{q} \star \tilde{r})), \delta_F(\tilde{p} \star \tilde{q})\} \leq s$. So $\tilde{p} \star \tilde{r} \in \ddot{\mathfrak{L}}(\delta_F; s)$. Hence, $\ddot{\mathfrak{L}}(\delta_F; s)$ is an IUPF of $\tilde{M}$.

For the converse, suppose that, $\ddot{\mathfrak{U}}(\omega_F; t)$ and $\ddot{\mathfrak{L}}(\delta_F; s)$ are IUPF of $\tilde{M}$ for each $t, s \in [0,1]$ such that $\ddot{\mathfrak{U}}(\omega_F; t) \neq \emptyset$ and $\ddot{\mathfrak{L}}(\delta_F; s) \neq \emptyset$. Let $\tilde{p} \in \tilde{M}$. Then we have $\tilde{p} \in \ddot{\mathfrak{U}}(\omega_F; \omega_F(\tilde{p}))$ and $\tilde{p} \in \ddot{\mathfrak{L}}(\delta_F; \delta_F(\tilde{p}))$. By assumption, we have $\ddot{\mathfrak{U}}(\omega; \omega_F(\tilde{p}))$ and $\ddot{\mathfrak{L}}(\delta_F; \delta_F(\tilde{p}))$ are IUPFs of $\tilde{M}$. Thus, $0 \in \ddot{\mathfrak{U}}(\omega_F; \omega_F(\tilde{p}))$ and $0 \in \ddot{\mathfrak{L}}(\delta_F; \delta_F(\tilde{p}))$ which imply $\omega_F(0) \geq \omega_F(\tilde{p})$ and $\delta_F(0) \leq \delta_F(\tilde{p})$. Suppose that there exist $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}$ such that $\omega_F(\tilde{p} \star \tilde{r}) < \min\{\omega_F(\tilde{p} \star (\tilde{q} \star \tilde{r})), \omega_F(\tilde{p} \star \tilde{q})\}$. Choose $t_0 = \frac{1}{2}[\omega_F(\tilde{p} \star \tilde{r}) + \min\{\omega_F(\tilde{p} \star (\tilde{q} \star \tilde{r})), \omega_F(\tilde{p} \star \tilde{q})\}]$. Thus, $t_0 \in [0,1]$ and $\omega_F(\tilde{p} \star \tilde{r}) < t_0 < \min\{\omega_F(\tilde{p} \star (\tilde{q} \star \tilde{r})), \omega_F(\tilde{p} \star \tilde{q})\}$. It implies that $\tilde{p} \star \tilde{r} \notin \ddot{\mathfrak{U}}(\omega_F; t_0)$ but

$\tilde{p}\star(\tilde{q}\star\tilde{r})$, $\tilde{p}\star\tilde{q} \in \ddot{\mathfrak{U}}(\omega_F; t_0)$. Thus, $\ddot{\mathfrak{U}}(\omega_F; t_0)$ is not an IUPF of $\tilde{M}$, which is a contradiction. Hence, $\omega_F(\tilde{p}\star\tilde{r}) \geq \min\{\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \omega_F(\tilde{p}\star\tilde{q})\}$ for all $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}$.

Similarly, there exist $\tilde{l}, \tilde{m}, \tilde{n} \in \tilde{M}$ such that $\delta_F\left(\tilde{l}\star\tilde{n}\right) > \max\left\{\delta_F\left(\tilde{l}\star(\tilde{m}\star\tilde{n})\right), \delta_F\left(\tilde{l}\star\tilde{m}\right)\right\}$. Choose $s_0 = \frac{1}{2}\left[\delta_F\left(\tilde{l}\star\tilde{n}\right) + \max\left\{\delta_F\left(\tilde{l}\star(\tilde{m}\star\tilde{n})\right), \delta_F\left(\tilde{l}\star\tilde{m}\right)\right\}\right]$. Thus, $s_0 \in [0,1]$ and $\max\left\{\delta_F\left(\tilde{l}\star(\tilde{m}\star\tilde{n})\right), \delta_F\left(\tilde{l}\star\tilde{m}\right)\right\} < s_0 < \delta_F\left(\tilde{l}\star\tilde{n}\right)$. It implies that $\tilde{l}\star\tilde{n} \notin \ddot{\mathfrak{L}}(\delta_F; s_0)$, but $\tilde{l}\star(\tilde{m}\star\tilde{n})$, $\tilde{l}\star\tilde{m} \in \ddot{\mathfrak{L}}(\delta_F; s_0)$. Thus, $\ddot{\mathfrak{L}}(\delta_F; s_0)$ is not an IUPF of $\tilde{M}$, which is a contradiction. Therefore, $\delta_F\left(\tilde{l}\star\tilde{n}\right) \leq \max\left\{\delta_F\left(\tilde{l}\star(\tilde{m}\star\tilde{n})\right), \delta_F\left(\tilde{l}\star\tilde{m}\right)\right\}$ for all $\tilde{l}, \tilde{m}, \tilde{n} \in \tilde{M}$. Therefore, $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$. □

**Corollary 3.1.** *An IFS $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$ if and only if for all $s, t \in [0,1]$, the $\ddot{\mathfrak{C}}(\omega_F, \delta_F; t, s)$ is either empty or an IUPF.*

**Proof:** The necessary condition is straightforward from Theorems 2.1 and 3.7.

Conversely, assume that the set $\ddot{\mathfrak{C}}(\omega_F, \delta_F; t, s)$ is either empty or an IUPF of $\tilde{M}$ for all $s, t \in [0,1]$. Let $t \in [0,1]$ be such that $\ddot{\mathfrak{U}}(\omega_F; t) \neq \emptyset$. Then $\emptyset \neq \ddot{\mathfrak{U}}(\omega_F; t) = \ddot{\mathfrak{U}}(\omega_F; t) \cap \tilde{M} = \ddot{\mathfrak{U}}(\omega_F; t) \cap \ddot{\mathfrak{L}}(\delta_F; 1) = \ddot{\mathfrak{C}}(\omega_F, \delta_F; t, 1)$. By assumption, we have $\ddot{\mathfrak{U}}(\omega_F; t) = \ddot{\mathfrak{C}}(\omega_F, \delta_F; t, 1)$ is an IUPF of $\tilde{M}$. Let $s \in [0,1]$ be such that $\ddot{\mathfrak{L}}(\delta_F; s) \neq \emptyset$. Then $\emptyset \neq \ddot{\mathfrak{L}}(\delta_F; s) = \tilde{M} \cap \ddot{\mathfrak{L}}(\delta_F; s) = \ddot{\mathfrak{U}}(\omega_F; 0) \cap \ddot{\mathfrak{L}}(\delta_F; s) = \ddot{\mathfrak{C}}(\omega_F, \delta_F; 0, s)$. By assumption, $\ddot{\mathfrak{L}}(\delta_F; s) = \ddot{\mathfrak{C}}(\omega_F, \delta_F; 0, s)$ is an IUPF of $\tilde{M}$. Hence, by Theorem 3.7, $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$. □

**Theorem 3.8.** *If an IFS $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$, then for all $s, t \in [0,1]$, the sets $\ddot{\mathfrak{U}}^+(\omega_F; t)$ and $\ddot{\mathfrak{L}}^-(\delta_F; s)$ are either empty or IUPFs of $\tilde{M}$.*

**Proof:** Suppose that an IFS $F = (\omega_F, \delta_F)$ is an IFIUPF of $\tilde{M}$. Let $s, t \in [0,1]$ be such that $\ddot{\mathfrak{U}}^+(\omega_F; t)$ and $\ddot{\mathfrak{L}}^-(\delta_F; s)$ are nonempty subsets of $\tilde{M}$. Then there exist $\tilde{a} \in \ddot{\mathfrak{U}}^+(\omega_F; t)$ and $\tilde{b} \in \ddot{\mathfrak{L}}^-(\delta_F; s)$ with $\omega_F(\tilde{a}) > t$ and $\delta_F\left(\tilde{b}\right) < s$. By assumption, we have $\omega_F(0) \geq \omega_F(\tilde{a}) > t$ and $\delta_F(0) \leq \delta_F\left(\tilde{b}\right) < s$, so $0 \in \ddot{\mathfrak{U}}^+(\omega_F; t)$ and $0 \in \ddot{\mathfrak{L}}(\delta_F; s)$. Let $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}$ be such that $\tilde{p}\star(\tilde{q}\star\tilde{r}), \tilde{p}\star\tilde{q} \in \ddot{\mathfrak{U}}^+(\omega_F; t)$. Then $\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})) > t$ and $\omega_F(\tilde{p}\star\tilde{q}) > t$. Thus, $\omega_F(\tilde{p}\star\tilde{r}) \geq \min\{\omega_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \omega_F(\tilde{p}\star\tilde{q})\} > t$, so $\tilde{p}\star\tilde{r} \in \ddot{\mathfrak{U}}^+(\omega_F; t)$. Hence, $\ddot{\mathfrak{U}}^+(\omega_F; t)$ is an IUPF of $\tilde{M}$. Finally, let $\tilde{p}, \tilde{q}, \tilde{r} \in \tilde{M}$ be such that $\tilde{p}\star(\tilde{q}\star\tilde{r}), \tilde{p}\star\tilde{q} \in \ddot{\mathfrak{L}}^-(\delta_F; s)$. Then $\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})) < s$ and $\delta_F(\tilde{p}\star\tilde{q}) < s$. Thus, $\delta_F(\tilde{p}\star\tilde{r}) \leq \max\{\delta_F(\tilde{p}\star(\tilde{q}\star\tilde{r})), \delta_F(\tilde{p}\star\tilde{q})\} < s$, so $\tilde{p}\star\tilde{r} \in \ddot{\mathfrak{L}}^-(\delta_F; s)$. Hence, $\ddot{\mathfrak{L}}^-(\delta_F; s)$ is an IUPF of $\tilde{M}$. □

4. **Conclusion.** In this paper, we have introduced the concept of IFIUPFs of UP-algebras and provided some properties of IFIUPFs and together studied the relation of IFIUPFs and IFUPFs in UP-algebras. We will further extend intuitionistic fuzzy comparative UP-filters in UP-algebras and study properties of intuitionistic fuzzy comparative UP-filters in UP-algebras.

In the near future, we will broaden the scope of the research covered in this work to include investigation into essential implicative UP-filters and $t$-essential intuitionistic fuzzy implicative UP-filters, in accordance with [22, 23].

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