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EXTENDED SEMI-STRONGLY STABILIZING CONTROLLER WITH A POLE AT THE ORIGIN

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Abstract. *A semi-strongly stabilizing controller has poles at the origin for the output of the control system to follow the step reference input without steady-state error. In some cases, the controller needs to have poles on the imaginary axis. Therefore, Kimura et al. proposed extended semi-strongly stabilizing controllers with poles on the imaginary axis and other poles in the open left-half plane. However, the controller proposed by Kimura et al. has no pole at the origin, and their system also has steady-state error for the reference input when the plant includes uncertainties and disturbances. In this paper, we overcome this problem and propose an extended semi-strongly stabilizing controller that has poles both at the origin and on the imaginary axis. In addition, we clarified the parameterization of all extended semi-strongly stabilizable plants and the parameterization of all extended semi-strongly stabilizing controllers for extended semi-strongly stabilizable plants.*

Keywords: Strongly stabilization, Semi-strongly stabilizing controllers, Parameterization

1. **Introduction.** Parameterization is a method of finding all stabilizing controllers for a given plant [1, 2]. By using parametrization, the stability of the control system is guaranteed. Various papers have been published on parameterization problems, such as proportional integral differential (PID) control [3], two-degree-of-freedom stabilizing controllers [4], disturbance observers [5], modified Smith predictors [6], and internally stabilizing controllers [7]. However, the stability of the controller obtained in these parametrizations is not taken into account. If the controller is unstable, the control system will be highly sensitive when parameters under control change [8].

To be minimally sensitive to parameter changes, stable controllers must be used. To address this problem, there exists a control method called strong stabilization, which is a method of stabilizing the control system by using stable controllers. Using this method means that there is no need to consider problems such as high sensitivity to disturbances and degradation of target-tracking performance, which occur when unstable controllers are used [9, 10]. However, all plants are not necessarily stabilized by stable controllers. A condition exists comprising strongly stabilizing controllers known as the parity interlacing

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property (pip.) condition [8, 11]; that is, the plant needs to satisfy the pip condition. Wakaiki et al. examine the sensitivity reduction problem with stable controllers for the linear time-invariant multi-input/multi-output distributed parameter system [17, 18]. However, they do not clarify the class of strongly stabilizable plants. If the class of strongly stabilizable plants is clarified, we can obtain the parameterization of all stable stabilizing controllers. In addition, we can clarify the characteristics of strongly stabilizable plants. From this viewpoint, Hoshikawa et al. clarify the class of all strongly stabilizable plants [12]. The reference in [13] clarifies the parameterization of all two-degree-of-freedom strongly stabilizing controllers.

Using strongly stabilizing controllers, when uncertainty in the plant exists or a step disturbance exists, the output of the control system cannot follow the step reference input without steady-state error. In many actual control systems, the output is required to follow the step reference input without steady-state error, even if uncertainty in the plant or the step disturbance exists. To overcome this problem, an integrator must be introduced to offset elimination from a set point. From this viewpoint, Hoshikawa et al. extended the concept of strong stabilization and proposed a concept of semi-strong stabilization, which is a stabilization by a controller that has a pole at the origin and the others in the open left-half plane [14]. Then, a class of semi-strongly stabilizable plants and a controller design method were proposed [15]. Using these controllers, a robust and reliable control system can be designed. However, the method by Hoshikawa et al. fails to place the poles on the imaginary axis. There exist control systems that need a pair of poles on the imaginary axis; for example, those with control systems to follow sinusoidal signals and self-repairing control systems for faulty sensors [16]. Therefore, Kimura et al. extended semi-strongly stabilizing controllers to have poles on the imaginary axis and other poles in the open left-half plane [19]. However, this extended semi-strong stabilizing controller does not have a pole at the origin, so this system could not eliminate the steady-state error from the output. The controller should have a pole at the origin to allow the output of the control system to follow the step reference input without steady-state error when uncertainty in the plant or a step disturbance exists.

In this paper, we define semi-stabilizing controllers with poles at the origin and on the imaginary axis as extended semi-strongly stabilizing controllers with a pole at the origin. We clarify parameterizations of all extended semi-strongly stabilizable plants and of all extended semi-strongly stabilizing controllers with a pole at the origin for extended semi-strongly stabilizable plants. This paper is organized as follows. In Section 2, the problem considered in this paper is explained. In Section 3, the class of all extended semi-strongly stabilizable plants is considered. That is, we clarify the parameterization of all extended semi-strongly stabilizable plants. In Section 4, for the extended semi-strongly stabilizable plant, the parameterization of all extended semi-strongly stabilizing controllers with a pole at the origin is clarified. In Section 5, we present a design method for extended semi-strongly stabilizing controllers with a pole at the origin. In Section 6, we show a numerical example and illustrate the effectiveness of the proposed method. Section 7 concludes.

2. **Problem Formulation.**

Consider the control system

$$\begin{cases} y(s) = G(s)u(s) + d(s) \\ u(s) = C(s)(r(s) - y(s)) \end{cases}, \tag{1}$$

where $G(s) \in R(s)$ is the plant, $C(s) \in R(s)$ is the controller, $y(s)$ is the output, $u(s)$ is the control input, $d(s)$ is the disturbance, and $r(s)$ is the references input.

Kimura et al. proposed extended semi-strongly controllers with poles on the imaginary axis [19]. However, that controller does not have a pole at the origin. The controller should have a pole at the origin to make the output of the control system follow the step reference input without steady-state error when uncertainty in the plant or a step disturbance exists. Therefore, we propose an extended semi-strongly stabilizing controller with poles at the origin and on the imaginary axis.

We define extended semi-strongly stabilizing controllers with a pole at the origin as follows.

Definition 2.1. *(Extended semi-strongly stabilizing controller with a pole at the origin) The controller $C(s)$ is called an extended semi-strongly stabilizing controller with a pole at the origin if the following expressions hold true.*

1) $C(s)$ makes the control system in (1) internally stable;

2) $C(s)$ has a pole at the origin and a pair of complex conjugate poles on the imaginary axis. The other poles of $C(s)$ are in the open left-half plane.

That is, if $C(s)$ in (1) written as

$$C(s) = \frac{Q_c(s)}{n_c(s)}, \tag{2}$$

stabilizes the control system in (1), we call the $C(s)$ in (1) the extended semi-strongly stabilizing controller at a pole at the origin, where $n_c \in RH_\infty$ is written as

$$n_c(s) = \frac{s\left(s^2+\omega^2\right)}{n_{cd}(s)}, \tag{3}$$

where $\omega \in R$ is any constant, $n_{cd}(s)$ is any Hurwitz polynomial of 3 degree, and $Q_c(s) \in RH_\infty$ is written as

$$Q_c(s)|_{s=0,\pm j\omega} \neq 0. \tag{4}$$

Note that all plants are not necessarily stabilized by extended semi-strongly stabilizing controllers with a pole at the origin. Therefore, we define extended semi-strongly stabilizable plants.

Definition 2.2. *(Extended semi-strongly stabilizable plant) When the plant $G(s)$ in (1) can be stabilized by the extended semi-strongly stabilizing controller $C(s)$ in (2), the plant $G(s)$ is called an extended semi-strongly stabilizable plant.*

The problem considered in this paper is to clarify the parameterizations of all extended semi-strongly stabilizable plants and of all extended semi-strongly stabilizing controllers with a pole at the origin.

3. **Parameterization of All Extended Semi-Strongly Stabilizable Plants.** In this section, we clarify the parameterization of all extended strongly stabilizable plants.

This parameterization is summarized in the following theorem.

Theorem 3.1. *The plant $G(s)$ is extended semi-strongly stabilizable if and only if the plant $G(s)$ is written as*

$$G(s) = \frac{n_b + n_c(s)Q_2(s)}{\frac{1-n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s)}, \tag{5}$$

where $n_b(s) \in RH_\infty$ is any function satisfying

$$1 - n_b(s)Q_1(s)|_{s=0,\pm j\omega} = 0, \tag{6}$$

$Q_2(s) \in RH_\infty$ is any function, and $Q_1(s) \in RH_\infty$ is any function satisfying

$$Q_1(s)|_{s=0,\pm j\omega} \neq 0. \tag{7}$$

The proof of Theorem 3.1 requires the following lemma.

Lemma 3.1. [8] *Assume that* $A(s) \in RH_\infty^{m\times n}$, $B(s) \in H_\infty^{q\times p}$, $C(s) \in RH_\infty^{m\times p}$ *and*

$$\mathrm{rank}\begin{bmatrix} A^T(s) & B^T(s) \end{bmatrix} = \gamma \tag{8}$$

are satisfied. There exist $X(s) \in RH_\infty$ *and* $Y(s) \in RH_\infty$ *satisfying*

$$X(s)A(s) + Y(s)B(s) = C(s) \tag{9}$$

if and only if there exists $U(s) \in \mathcal{U}$ *satisfying*

$$\begin{bmatrix} A(s) \\ B(s) \\ C(s) \end{bmatrix} = U(s)\begin{bmatrix} A(s) \\ B(s) \\ O \end{bmatrix}. \tag{10}$$

When $X_0(s) \in RH_\infty$ *and* $Y_0(s) \in RH_\infty$ *are solutions to* (9), *then all solutions to* (9) *are given by*

$$\begin{bmatrix} X(s) & Y(s) \end{bmatrix} = \begin{bmatrix} X_0(s) & Y_0(s) \end{bmatrix} + Q(s)\begin{bmatrix} W_1(s) & W_2(s) \end{bmatrix}, \tag{11}$$

where $W_1(s)$ *and* $W_2(s)$ *satisfy*

$$W_1(s)A(s) + W_2(s)B(s) = 0 \tag{12}$$

and

$$\mathrm{rank}\begin{bmatrix} W_1(s) & W_2(s) \end{bmatrix} = n + q - \gamma \tag{13}$$

and $Q(s) \in RH_\infty^{p\times(n+q-\gamma)}$ *is any function.*

Using Lemma 3.1, we will prove Theorem 3.1.

Proof: First, the necessity is shown. That is, we show that if $C(s)$ in (2) makes the control system in (1) stable, then $G(s)$ takes the form of (5). From the assumption that $C(s)$ in (2) stabilizes the plant $G(s)$,

$$N(s)Q_c(s) + D(s)n_c(s) = 1, \tag{14}$$

where $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ are coprime factors of $G(s)$ satisfying

$$G(s) = \frac{N(s)}{D(s)} \tag{15}$$

and $Q_c(s) \in RH_\infty$ and $n_c(s)$ are coprime factors of $C(s)$ in (2). From the assumption that $C(s)$ in (2) makes the control system in (1) stable, $N(s)N_c(s) + D(s)D_c(s) \in \mathcal{U}$, that is, from (2) and (15),

$$N(s)N_c(s) + D(s)D_c(s) = N(s)Q_1(s) + D(s)n_c(s) = 1, \tag{16}$$

where $Q_1(s) \in RH_\infty$ is an arbitrary function satisfying (7). A pair of solutions $N_0(s)$ and $D_0(s)$ to (16) is

$$N_0(s) = n_b \tag{17}$$

and

$$D_0(s) = \frac{1 - n_b(s)Q_1(s)}{n_c(s)}. \tag{18}$$

From Lemma 3.1, all solutions $N(s)$ and $D(s)$ satisfying (16) are written as

$$N(s) = n_b + n_c(s)Q_2(s) \tag{19}$$

and

$$D(s) = \frac{1 - n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s), \tag{20}$$

because

$$n_b(s)Q_1(s) + \frac{1 - n_b(s)Q_1(s)}{n_c(s)}n_c(s) = 1 \tag{21}$$

and

$$n_c(s)Q_1(s) - Q_1(s)n_c(s) = 0, \tag{22}$$

respectively, where $n_b(s)$ satisfies (6) and $Q_2(s) \in RH_\infty$ is any function. Substituting (19) and (20) for (15), we have (5). Thus, the necessity has been shown.

Next, sufficiency is shown. That is, if $G(s)$ in (1) takes the form of (5), then there exists an extended semi-strongly stabilizing controller to make the control system in (1) stable. A controller is set as

$$C(s) = \frac{Q_1(s)}{n_c(s)}. \tag{23}$$

From simple manipulation and (23), we have

$$\frac{G(s)C(s)}{1+G(s)C(s)} = Q_1(s)\left(n_b(s) + n_c(s)Q_2(s)\right), \tag{24}$$

$$\frac{G(s)}{1+G(s)C(s)} = n_c(s)\left(n_b(s) + n_c(s)Q_2(s)\right), \tag{25}$$

$$\frac{C(s)}{1+G(s)C(s)} = Q_1(s)\left(\frac{1-n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s)\right) \tag{26}$$

and

$$\frac{1}{1+G(s)C(s)} = n_c(s)\left(\frac{1-n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s)\right), \tag{27}$$

because $n_c(s) \in RH_\infty$, $n_b(s) \in RH_\infty$, $Q_1(s) \in RH_\infty$, and $Q_2(s) \in RH_\infty$, (24), (25), (26) and (27) are stable. Thus, sufficiency has been shown.

We have thus proved Theorem 3.1. □

4. **Parameterization of All Extended Semi-Strongly Stabilizing Controllers.** In this section, the parameterization of all extended semi-strongly stabilizing controllers $C(s)$ for the extended semi-strongly stabilizable plant $G(s)$ in (5) is proposed.

This parameterization is summarized as the following theorem.

Theorem 4.1. *The controller $C(s)$ is an extended semi-strongly stabilizing controller with a pole at the origin for the plant $G(s)$ in (5) if and only if the controller $C(s)$ is written as*

$$C(s) = \frac{Q_1(s) + \left\{\frac{1-n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s)\right\}P(s)}{n_c(s) - \left\{n_b + n_c(s)Q_2(s)\right\}P(s)}, \tag{28}$$

where $P(s) \in RH_\infty$ and $Q(s) \in RH_\infty$ are functions written as

$$P(s) = n_c(s)Q(s), \tag{29}$$

$$Q(s) = \frac{1-\hat{Q}(s)}{n_b + n_c(s)Q_2(s)}, \tag{30}$$

respectively, $\hat{Q}(s) \in \mathcal{U}$ is a unimodular function that makes $Q(s)$ proper and satisfies

$$\left.\frac{1}{(s-s_i)^{m_i-1}}\left\{1-\hat{Q}(s)\right\}\right|_{s=s_i} = 0 \ \forall i, \tag{31}$$

$s_i \in R$ are unstable zeros of $n_b(s) + n_c(s)Q_2(s)$, and m_i are the multiplicity.

Proof: From [8], the parameterization of all stabilizing controllers for $G(s)$ is written as

$$C(s) = \frac{X(s) + D(s)P(s)}{Y(s) - N(s)P(s)}, \tag{32}$$

when $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ are coprime factors of $G(s)$ on RH_∞ satisfying

$$G(s) = \frac{N(s)}{D(s)}, \tag{33}$$

$X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ are the solutions of

$$N(s)X(s) + D(s)Y(s) = 1, \tag{34}$$

and $P(s) \in RH_\infty$ is any function. Because the extended semi-strongly stabilizable plant $G(s)$ takes the form of (5), coprime factors $N(s)$ and $D(s)$ in (33) are written as

$$N(s) = n_b + n_c(s)Q_2(s) \tag{35}$$

and

$$D(s) = \frac{1 - n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s). \tag{36}$$

From (35) and (36), a pair of $X(s)$ and $Y(s)$ satisfying (34) is given by

$$X(s) = Q_1(s) \tag{37}$$

and

$$Y(s) = n_c(s). \tag{38}$$

Substituting (35), (36), (37), and (38) for (32), we have (28) where $P(s) \in RH_\infty$ is any function.

We show necessity. That is, we show that if $C(s)$ in (28) is an extended semi-strongly stabilizing controller, then $P(s)$ in (28) is given by (29), $Q(s)$ in (29) is given by (30), and $\hat{Q}(s)$ in (30) satisfies $\hat{Q}(s) \in \mathcal{U}$ and (31). From the assumption that $C(s)$ in (28) is an extended semi-strongly stabilizing controller,

$$n_c(s) - \{n_b + n_c(s)Q_2(s)\} P(s)|_{s=0,\pm j\omega} = 0 \tag{39}$$

is satisfied. Because $n_b(s)|_{s=0,\pm j\omega} \neq 0$, this equation yields

$$P(s)|_{s=0,\pm j\omega} = 0. \tag{40}$$

The equation implies that $P(s)$ is given by (29), where $Q(s) \in RH_\infty$. Substituting (29) for (28), (28) is then written as

$$C(s) = \frac{1}{n_c(s)} \left\{ Q_1(s) + \frac{Q(s)}{1 - (n_b + n_c(s)Q_2(s))\, Q(s)} \right\}. \tag{41}$$

From the assumption that $C(s)$ in (2) is an extended semi-strongly stabilizing controller,

$$n_c(s)C(s) = Q_1(s) + \frac{Q(s)}{1 - (n_b + n_c(s)Q_2(s))\, Q(s)} \in RH_\infty \tag{42}$$

must be satisfied because $Q_1(s) \in RH_\infty$ and $Q(s) \in RH_\infty$,

$$1 - (n_b + n_c(s)Q_2(s))\, Q(s) \in \mathcal{U}. \tag{43}$$

Using any function $\hat{Q}(s) \in \mathcal{U}$, (43) is then written as

$$\hat{Q}(s) = 1 - (n_b + n_c(s)Q_2(s))\, Q(s). \tag{44}$$

Equation (44) corresponds to (30). Because $Q(s) \in RH_\infty$,

$$\frac{1}{(s - s_i)^{m_i - 1}} \{n_b + n_c(s)Q_2(s)\} Q(s)|_{s=s_i} = 0 \ \ \forall i \tag{45}$$

is satisfied, when s_i are unstable zeros of $n_b + n_c(s)Q_2(s)$ and multiplicities of s_i are m_i. From (44) and (45), (31) is satisfied. Thus, the necessity has been shown.

Next, we show sufficiency. That is, we show that if $P(s)$ in (28) is given by (29), $Q(s)$ in (29) is given by (30), and $\hat{Q}(s)$ in (30) satisfies $\hat{Q}(s) \in \mathcal{U}$ and (31), then $C(s)$ in (28) is an extended semi-strongly stabilizing controller. Substituting (29) and (30) for (28), we have

$$C(s) = \frac{1}{n_c(s)}\left\{Q_1(s) + \frac{1-\hat{Q}(s)}{(n_b + n_c(s)Q_2(s))\,\hat{Q}(s)}\right\} = \frac{1}{n_c(s)}\left\{Q_1(s) + \frac{Q(s)}{\hat{Q}(s)}\right\}. \tag{46}$$

From (46), $Q_1(s) \in RH_\infty$, $\hat{Q}(s) \in \mathcal{U}$, and $Q(s) \in RH_\infty$, the controller $C(s)$ in (28) has a pole at the origin and a pair of complex conjugate poles on the imaginary axis and the other poles of $C(s)$ are in the closed left-half plane. Next, we show that $C(s)$ in (46) makes the control system in (1) stable. By simple manipulation, we have

$$\frac{G(s)C(s)}{1+G(s)C(s)} = 1 - n_c(s)\hat{Q}(s)\left(\frac{1-n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s)\right), \tag{47}$$

$$\frac{G(s)}{1+G(s)C(s)} = n_c(s)(n_b(s) + n_c(s)Q_2(s))\hat{Q}(s), \tag{48}$$

$$\frac{C(s)}{1+G(s)C(s)} = \left(\frac{1-n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s)\right)\left(Q_1(s)\hat{Q}(s) + Q(s)\right) \tag{49}$$

and

$$\frac{1}{1+G(s)C(s)} = n_c(s)\left(\frac{1-n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s)\right)\hat{Q}(s). \tag{50}$$

Because $n_c(s) \in RH_\infty$, $n_b(s) \in RH_\infty$, $Q_2(s) \in RH_\infty$, $\hat{Q}(s) \in \mathcal{U}$, and $Q(s) \in RH_\infty$, the transfer functions in (47), (48), (49), and (50) are stable. This implies that the control system in (1) is stable. Thus, sufficiency has been shown.

We have thus proved Theorem 4.1. □

5. **Design Method.** In this section, we present a design method for an extended semi-strongly stabilizing controller $C(s)$ with a pole at the origin in (28).

From Theorem 4.1, designing an extended semi-strongly stabilizing controller $C(s)$ requires designing $\hat{Q}(s) \in \mathcal{U}$ that satisfies (31) and makes $Q(s)$ in (30) proper. A design method for $\hat{Q}(s)$ is summarized as follows.

1) We factor

$$n_b(s) + n_c(s)Q_2(s) = \tilde{Q}(s) \tag{51}$$

as

$$n_b(s) + n_c(s)Q_2(s) = \tilde{Q}_i(s)\tilde{Q}_o(s), \tag{52}$$

where $\tilde{Q}_i(s) \in RH_\infty$ is an inner function and $\tilde{Q}_o(s) \in RH_\infty$ is an outer function.

2) Using $\tilde{Q}_o(s) \in RH_\infty$, we set $\bar{Q}(s) \in RH_\infty$ as

$$\bar{Q}(s) = \frac{q(s)}{\tilde{Q}_o(s)}, \tag{53}$$

where

$$q(s) = \frac{k}{(\rho s + 1)^m}, \tag{54}$$

$\rho \in R$ is any positive number, m is any positive integer to make $\bar{Q}(s)$ proper, and $k \in R$ is a real number that satisfies $0 < k < 1$, and

$$k \simeq 1. \tag{55}$$

3) $\hat{Q} \in \mathcal{U}$ is designed as

$$\hat{Q}(s) = 1 - (n_b(s) + n_c(s)Q_2(s))\bar{Q}(s). \tag{56}$$

Next, we confirm that $\hat{Q}(s)$ in (56) satisfies $\hat{Q}(s) \in \mathcal{U}$, (31) and makes $Q(s)$ in (30) proper. First, we show that $\hat{Q}(s)$ in (56) satisfies $\hat{Q}(s) \in \mathcal{U}$. Substituting (52) and (53) for (56), $\hat{Q}(s)$ in (56) is then written as

$$\hat{Q}(s) = 1 - \tilde{Q}_i(s)q(s). \tag{57}$$

Because $\tilde{Q}_i(s)$ is an inner function, $\tilde{Q}_i(s)$ is biproper. Thus, $\tilde{Q}_i(s)q(s)$ is proper. In addition, from (54) and $0 < k < 1$,

$$\left\| \tilde{Q}_i(s)q(s) \right\|_\infty < 1. \tag{58}$$

This implies that $\hat{Q} \in \mathcal{U}$. Next, we show that (31) is satisfied. Because s_i are unstable zeros of $n_b(s) + n_c(s)Q_2(s)$, m_i denote the multiplicities of s_i, $\tilde{Q}_i$ is an inner function of $n_b(s) + n_c(s)Q_2(s)$, and (54),

$$\left. \frac{1}{(s-s_i)^{m_i-1}} \tilde{Q}_i(s)q(s) \right|_{s=s_i} = 0 \ \ \forall i \tag{59}$$

is satisfied. From (57) and (59), (31) is satisfied. Next, we show $\hat{Q}(s)$ in (56) makes $Q(s)$ in (30) proper. Substituting (56) for (30),

$$Q(s) = \bar{Q}(s) \tag{60}$$

is satisfied. Because $\bar{Q}(s) \in RH_\infty$, $\hat{Q}(s)$ in (56) makes $Q(s)$ in (30) proper. Thus, we have shown that $\hat{Q}(s)$ in (56) satisfies $\hat{Q}(s) \in \mathcal{U}$, (31) and makes $Q(s)$ in (30) proper.

Therefore, we can design an extended semi-strongly stabilizing controller $C(s)$ using obtained $\hat{Q}(s)$.

6. **Numerical Example.** In this section, we illustrate a numerical example to show the effectiveness of the proposed parameterization of all extended semi-strongly stabilizing controllers with a pole at the origin.

Consider the problem of designing an extended semi-strongly stabilizing controller $C(s)$ for the plant $G(s)$, written as

$$G(s) = \frac{14(s+0.2725)(s^2+0.8704s+1.049)}{(s-2)(s+1)^3}. \tag{61}$$

The plant $G(s)$ is set to be unstable to show that the controller $C(s)$ stabilizes the control system when the plant $G(s)$ is unstable, because $G(s)$ is written as the form of (5) where

$$n_c(s) = \frac{s(s^2+1)}{(s+1)^3}, \tag{62}$$

$$n_b(s) = 2, \tag{63}$$

$$Q_1(s) = 0.5, \tag{64}$$

and

$$Q_2(s) = \frac{-2(s-2)}{s+2}. \tag{65}$$

Then, $n_c(s) \in RH_\infty$ satisfies (3), $n_b(s) \in RH_\infty$ satisfies (6), $Q_1(s) \in RH_\infty$ satisfies (7), and $Q_2(s) \in RH_\infty$. Therefore, $G(s)$ in (61) is an extended semi-strongly stabilizable plant.

Next, we design an extended semi-strongly stabilizing controller $C(s)$ using the method in Section 5. $\tilde{Q}(s)$ is factored by (52), where

$$\tilde{Q}_i(s) = 1 \tag{66}$$

and

$$\tilde{Q}_o(s) = \frac{14(s+0.2725)(s^2+0.8704s+1.049)}{(s+2)(s+1)^3}, \tag{67}$$

respectively. Then, $\tilde{Q}_i(s) \in RH_\infty$ is an inner function, and $\tilde{Q}_o(s) \in RH_\infty$ is an outer function. $\bar{Q}(s)$ is settled by (53), where $q(s)$ is written as (54),

$$\rho = 2, \tag{68}$$

$$m = 1, \tag{69}$$

and

$$k = 0.99. \tag{70}$$

Then, $\rho \in R$ is a positive number, m is a positive integer to make $\bar{Q}(s)$ proper, and $k \in R$ is a real number that satisfies $0 < k < 1$ and satisfies (55). Using this $\bar{Q}(s)$, $\hat{Q}(s)$ is written as (56). Thus, $\hat{Q}(s) \in \mathcal{U}$ becomes

$$\hat{Q} = \frac{s + 0.005}{s + 0.5}. \tag{71}$$

Substituting (71) for (28), we have

$$C(s) = \frac{0.53536(s+1)^3(s^2 + 0.5182s + 0.1081)(s^2 + 0.8841s + 1.234)}{s(s^2+1)(s + 0.2725)(s + 0.005)(s^2 + 0.8704s + 1.049)}. \tag{72}$$

$C(s)$ in (72) has unstable poles at the origin and $\pm j$. If $C(s)$ in (72) stabilizes the control system in (1), then $C(s)$ in (72) is an extended semi-strongly stabilizing controller.

Using $C(s)$ in (72), the response of the output $y(t)$ of the control system in (1) for the reference input $r(t) = 1$ is shown in Figure 1. Figure 1 shows that the control system in (1) is stable and that the output $y(t)$ follows the step reference input $r(t) = 1$ without steady-state error.

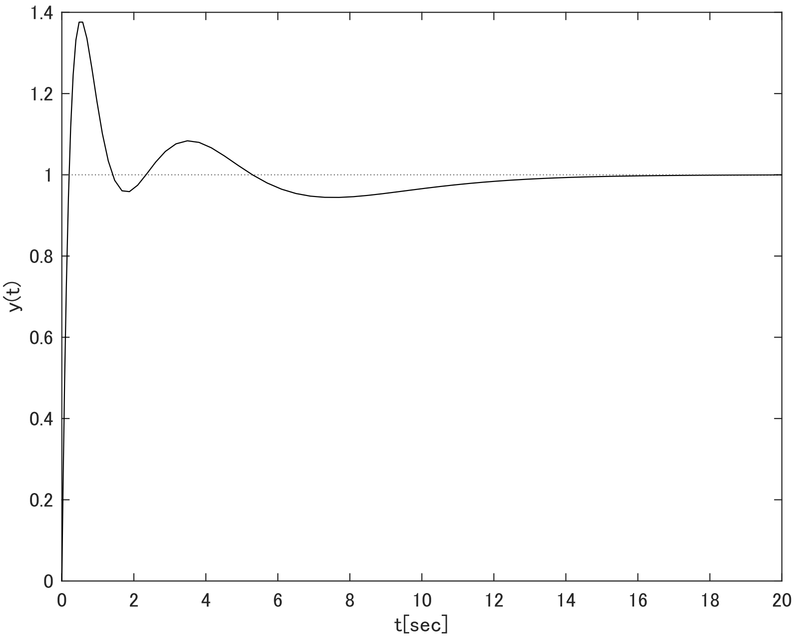

FIGURE 1. Response of the output $y(t)$ for $r(t) = 1$

The response of the output $y(t)$ of the control system in (1) for the sinusoidal disturbance $d(t) = \sin t$ is shown in Figure 2. Figure 2 shows that the sinusoidal disturbance $d(t) = \sin t$ is effectively attenuated.

7. **Conclusion.** In this paper, we have clarified parameterizations of all extended semi-strongly stabilizable plants and of all extended semi-strongly stabilizing controllers with a pole at the origin. We present a design method for extended semi-strongly stabilizing controllers with a pole at the origin. We show a numerical example and have illustrated the effectiveness of the proposed method. In the future, we consider introducing the extended semi-strongly stabilizing controller proposed in this paper to the fault detection method using resonance.

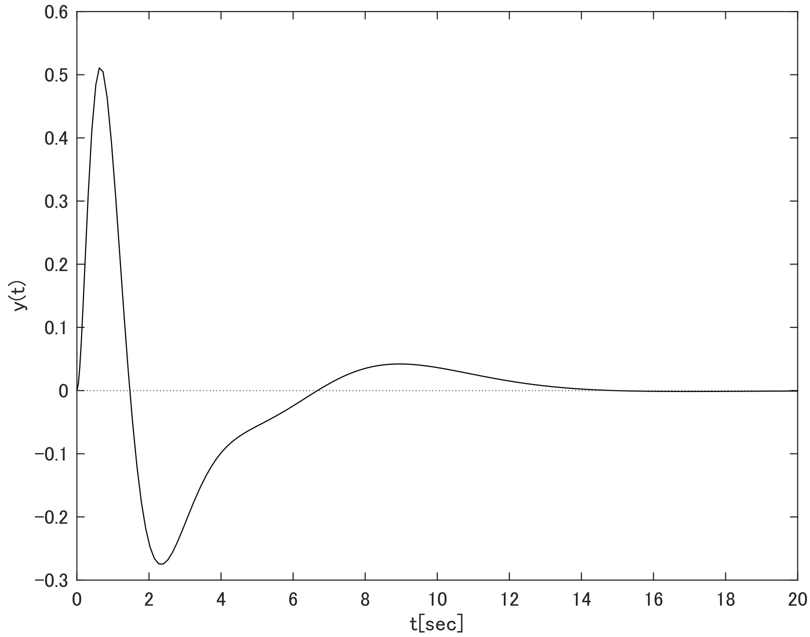

FIGURE 2. Response of the output for $d(t) = \sin t$

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