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ANTI-FUZZY SUBALGEBRAS/IDEALS/DEDUCTIVE SYSTEMS OF HILBERT ALGEBRAS

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Abstract. *In this paper, the concepts of anti-fuzzy subalgebras (AF subalgebras), anti-fuzzy ideals (AF ideals), and anti-fuzzy deductive systems (AF deductive systems) of Hilbert algebras are introduced and proved of some results. We discuss the relation between AF subalgebras (resp., AF ideals, AF deductive systems) and their level subsets. AF subalgebras, AF ideals, and AF deductive systems are also applied in the Cartesian product of Hilbert algebras. We also introduce the notion of the Cartesian product of fuzzy sets, and then we study related properties.*
Keywords: Hilbert algebra, Anti-fuzzy subalgebra, Anti-fuzzy ideal, Anti-fuzzy deductive system

1. **Introduction.** The concept of fuzzy sets was proposed by Zadeh [1]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets, rough sets, hybrid structures, and tripolar fuzzy sets has been discussed in [2, 3, 4, 5, 6]. The concept of Hilbert algebras was introduced in early 50-ties by Henkin [7] for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by Diego [8] from algebraic point of view. Diego [8] proved that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by Busneag [9, 10] and Jun [11] and some of their filters forming deductive systems were recognized. Dudek [12] considered the fuzzification of subalgebras/ideals and deductive systems in Hilbert algebras. The concept of fuzzy sets in anti-type has been studied in several algebras, which can be seen in [13, 14, 15], which inspired our study in this paper.

In this paper, we present new concepts of fuzzy subalgebras, fuzzy ideals, and fuzzy deductive systems of Hilbert algebras in anti-type and we call them AF subalgebras, AF ideals, and AF deductive systems, and show several results related to them. The

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relationship between AF subalgebras (resp., AF ideals, AF deductive systems) and their level subsets is discussed. In the Cartesian product of Hilbert algebras, AF subalgebras, AF ideals, and AF deductive systems are also applied. We also define the Cartesian product of fuzzy sets and investigate its features.

2. **Preliminaries.** Before we begin, let us go through the concept of Hilbert algebras as described by Diego [8] in 1966.

Definition 2.1. [8] *A Hilbert algebra is a triplet $X = (X, \cdot, 1)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed element of X such that the following axioms hold:*

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(3) $(\forall x, y \in X)(x \cdot y = 1,\ y \cdot x = 1 \Rightarrow x = y)$.

The following result was proved in [12].

Lemma 2.1. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x \cdot x = 1)$,
(2) $(\forall x \in X)(1 \cdot x = x)$,
(3) $(\forall x \in X)(x \cdot 1 = 1)$,
(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.2. [16] *A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a subalgebra of X if $x \cdot y \in D$ for all $x, y \in D$.*

Definition 2.3. [17] *A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called an ideal of X if the following conditions hold:*

(1) $1 \in D$,
(2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$,
(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

Definition 2.4. [18] *A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a deductive system of X if*

(1) $1 \in D$,
(2) $(\forall x, y \in X)(x, x \cdot y \in D \Rightarrow y \in D)$.

A *fuzzy set* [1] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.5. [19] *A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a fuzzy subalgebra of X if the following condition holds:*

$$(\forall x, y \in X)(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 2.6. [20] *A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a fuzzy ideal of X if the following conditions hold:*

(1) $(\forall x \in X)(\mu(1) \geq \mu(x))$,
(2) $(\forall x, y \in X)(\mu(x \cdot y) \geq \mu(y))$,
(3) $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu(y_1), \mu(y_2)\})$.

Definition 2.7. [12] *A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a fuzzy deductive system of X if the following conditions hold:*

(1) $(\forall x \in X)(\mu(1) \geq \mu(x))$,
(2) $(\forall x, y \in X)(\mu(y) \geq \min\{\mu(x \cdot y), \mu(x)\})$.

3. **Main Results.** In this section, we introduce the concepts of AF subalgebras/ideals/deductive systems of Hilbert algebras and investigate some related properties.

Definition 3.1. *A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be an anti-fuzzy subalgebra (AF subalgebra) of X if the following condition holds:*

$$(\forall x, y \in X)(\mu(x \cdot y) \leq \max\{\mu(x), \mu(y)\}).$$

Proposition 3.1. *Every AF subalgebra μ of a Hilbert algebra $X = (X, \cdot, 1)$ satisfies $\mu(1) \leq \mu(x)$ for all $x \in X$.*

Proof: For any $x \in X$, we have $\mu(1) = \mu(x \cdot x) \leq \max\{\mu(x), \mu(x)\} = \mu(x)$. □

Proposition 3.2. *If a fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is an AF subalgebra, then*

$$(\forall x \in X)(\mu(1 \cdot x) \leq \mu(x)).$$

Proof: For any $x \in X$, we have

$$\begin{aligned} \mu(1 \cdot x) &\leq \max\{\mu(1), \mu(x)\} \\ &= \max\{\mu(x \cdot x), \mu(x)\} \\ &\leq \max\{\max\{\mu(x), \mu(x)\}, \mu(x)\} \\ &= \mu(x). \end{aligned}$$

□

The subset $\{x \in X : \mu(x) = \mu(1)\}$ of a Hilbert algebra $X = (X, \cdot, 1)$ is denoted by μ^1.

Theorem 3.1. *If μ is an AF subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$, then the set μ^1 is a subalgebra of X.*

Proof: Let $x, y \in \mu^1$. Then $\mu(x) = \mu(1) = \mu(y)$. Thus, $\mu(x \cdot y) \leq \max\{\mu(x), \mu(y)\} = \mu(1)$. By using Proposition 3.1, we have $\mu(x \cdot y) = \mu(1)$, that is, $x \cdot y \in \mu^1$. Hence, μ^1 is a subalgebra of X. □

Proposition 3.3. *Let μ be a fuzzy set of a Hilbert algebra $X = (X, \cdot, 1)$. Then μ is an AF subalgebra of X if and only if $\overline{\mu}$ is a fuzzy subalgebra of X, where the fuzzy set $\overline{\mu}$ is defined by $\overline{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.*

Proof: Straightforward. □

Definition 3.2. *A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be an anti-fuzzy ideal (AF ideal) of X if the following conditions hold:*
(1) $(\forall x \in X)(\mu(1) \leq \mu(x))$,
(2) $(\forall x, y \in X)(\mu(x \cdot y) \leq \mu(y))$,
(3) $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{\mu(y_1), \mu(y_2)\})$.

Proposition 3.4. *Let μ be a fuzzy set of a Hilbert algebra $X = (X, \cdot, 1)$. Then μ is an AF ideal of X if and only if $\overline{\mu}$ is a fuzzy ideal of X.*

Proof: Straightforward. □

Proposition 3.5. *If μ is an AF ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)(\mu((y \cdot x) \cdot x) \leq \mu(y)).$$

Proof: Let $x, y \in X$. Then $\mu((y \cdot x) \cdot x) = \mu((y \cdot (1 \cdot x)) \cdot x) \leq \max\{\mu(y), \mu(1)\} = \mu(y)$. □

Lemma 3.1. *If μ is an AF ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)(x \leq y \Rightarrow \mu(x) \geq \mu(y)).$$

Proof: Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 1$ and so

$$\begin{aligned} \mu(y) &= \mu(1 \cdot y) \\ &= \mu(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\leq \max\{\mu(x \cdot y), \mu(x)\} \\ &= \max\{\mu(1), \mu(x)\} \\ &= \mu(x). \end{aligned}$$

□

Theorem 3.2. *Every AF ideal of a Hilbert algebra $X = (X, \cdot, 1)$ is an AF subalgebra of X.*

Proof: Let μ be an AF ideal of X. Let $x, y \in X$. Then $\mu(x \cdot y) \leq \mu(y) \leq \max\{\mu(x), \mu(y)\}$. Hence, H is an AF subalgebra of X. □

Definition 3.3. *A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be an anti-fuzzy deductive system (AF deductive system) of X if the following conditions hold:*

(1) $(\forall x \in X)(\mu(1) \leq \mu(x))$,
(2) $(\forall x, y \in X)(\mu(y) \leq \max\{\mu(x \cdot y), \mu(x)\})$.

Proposition 3.6. *Every AF ideal of a Hilbert algebra $X = (X, \cdot, 1)$ is an AF deductive system of X.*

Proof: Let μ be an AF ideal of X. Let $x, y \in X$. By Lemma 2.1, we have

$$\mu(y) = \mu(1 \cdot y) = \mu(((x \cdot y) \cdot (x \cdot y)) \cdot y) \leq \max\{\mu(x \cdot y), \mu(x)\}.$$

Hence, μ is an AF deductive system of X. □

Definition 3.4. *Let μ be a fuzzy set in a nonempty set X. For any $t \in [0, 1]$, the sets*

$$\begin{aligned} &U(\mu, t) = \{x \in X : \mu(x) \geq t\}, \\ &U^+(\mu, t) = \{x \in X : \mu(x) > t\}, \\ &L(\mu, t) = \{x \in X : \mu(x) \leq t\}, \\ &L^-(\mu, t) = \{x \in X : \mu(x) < t\}, \\ &E(\mu, t) = \{x \in X : \mu(x) = t\}. \end{aligned}$$

The following lemma is easy to prove.

Lemma 3.2. *Let μ be a fuzzy set in a nonempty set X. For any $t \in [0, 1]$, the following properties hold:*

(1) $L(\mu, t) = U(\mu, 1 - t)$,
(2) $L^-(\mu, t) = U^+(\mu, 1 - t)$,
(3) $U(\mu, t) = L(\mu, 1 - t)$,
(4) $U^+(\mu, t) = L^-(\mu, 1 - t)$.

Theorem 3.3. *Let μ be a fuzzy set in a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold:*

(1) μ is an AF subalgebra of X if and only if it satisfies for all $t \in [0, 1]$, $L(\mu, t) \neq \emptyset$ implies $L(\mu, t)$ is a subalgebra of X,
(2) μ is an AF subalgebra of X if and only if it satisfies for all $t \in [0, 1]$, $L^-(\mu, t) \neq \emptyset$ implies $L^-(\mu, t)$ is a subalgebra of X,
(3) $\overline{\mu}$ is an AF subalgebra of X if and only if it satisfies for all $t \in [0, 1]$, $U(\mu, t) \neq \emptyset$ implies $U(\mu, t)$ is a subalgebra of X,

(4) $\overline{\mu}$ *is an AF subalgebra of* X *if and only if it satisfies for all* $t \in [0,1]$, $U^+(\mu,t) \neq \emptyset$ *implies* $U^+(\mu,t)$ *is a subalgebra of* X.

Proof: (1) Let μ be an AF subalgebra of X and $t \in [0,1]$ be such that $L(\mu,t) \neq \emptyset$. Let $x,y \in X$ be such that $x,y \in L(\mu,t)$. Then $\mu(x) \leq t$ and $\mu(y) \leq t$. Thus, $\mu(x \cdot y) \leq \max\{\mu(x),\mu(y)\} \leq t$, so $x \cdot y \in L(\mu,t)$. Hence, $L(\mu,t)$ is a subalgebra of X.

Conversely, assume that every nonempty set $L(\mu,t)$ is a subalgebra in X. Let $x,y \in X$ and let $t = \max\{\mu(x),\mu(y)\}$, so $t \in [0,1]$. Thus, $\mu(x) \leq t$ and $\mu(y) \leq t$, so $x,y \in L(\mu,t)$. Thus, $L(\mu,t) \neq \emptyset$. By assumption, we have $L(\mu,t)$ is a subalgebra of X. Thus, $x \cdot y \in L(\mu,t)$, so $\mu(x \cdot y) \leq t = \max\{\mu(x),\mu(y)\}$. Hence, μ is an AF subalgebra of X.

(2) Let μ be an AF subalgebra of X and $t \in [0,1]$ be such that $L^-(\mu,t) \neq \emptyset$. Let $x,y \in X$ be such that $x,y \in L^-(\mu,t)$. Then $\mu(x) < t$ and $\mu(y) < t$. Thus, $\mu(x \cdot y) \leq \max\{\mu(x),\mu(y)\} < t$, so $x \cdot y \in L^-(\mu,t)$. Hence, $L^-(\mu,t)$ is a subalgebra of X.

Conversely, assume that every nonempty set $L^-(\mu,t)$ is a subalgebra in X. Let $x,y \in X$ be such that $\mu(x \cdot y) > \max\{\mu(x),\mu(y)\}$. Then $\mu(x \cdot y) \in [0,1]$. Choose $t = \mu(x \cdot y)$. Thus, $\mu(x) < t$ and $\mu(y) < t$, so $x,y \in L^-(\mu,t) \neq \emptyset$. By assumption, we have $L^-(\mu,t)$ is a subalgebra of X and so $x \cdot y \in L^-(\mu,t)$. Then $\mu(x \cdot y) < t = \mu(x \cdot y)$, which is a contradiction. Hence, $\mu(x \cdot y) \leq \max\{\mu(x),\mu(y)\}$ for all $x,y \in X$. Therefore, μ is an AF subalgebra of X.

(3) Let $\overline{\mu}$ be an AF subalgebra of X and $t \in [0,1]$ be such that $U(\mu,t) \neq \emptyset$. Let $x,y \in U(\mu,t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus, $\overline{\mu}(x \cdot y) \leq \max\{\overline{\mu}(x),\overline{\mu}(y)\}$. Then $1-\mu(x \cdot y) \leq \max\{1-\mu(x),1-\mu(y)\} = 1-\min\{\mu(x),\mu(y)\}$. Thus, $\mu(x \cdot y) \geq \min\{\mu(x),\mu(y)\} \geq t$, so $x \cdot y \in U(\mu,t)$. Hence, $U(\mu,t)$ is a subalgebra of X.

Conversely, assume that every nonempty set $U(\mu,t)$ is a subalgebra of X. Let $x,y \in X$ and $t = \min\{\mu(x),\mu(y)\}$, so $t \in [0,1]$. Thus, $\mu(x) \geq t$ and $\mu(y) \geq t$, so $x,y \in U(\mu,t) \neq \emptyset$. By assumption, we have $U(\mu,t)$ is a subalgebra of X and so $x \cdot y \in U(\mu,t)$. Then $\mu(x \cdot y) \geq t = \min\{\mu(x),\mu(y)\}$. Thus, $\overline{\mu}(x \cdot y) = 1-\mu(x \cdot y) \leq 1-\min\{\mu(x),\mu(y)\} = \max\{1-\mu(x),1-\mu(y)\} = \max\{\overline{\mu}(x),\overline{\mu}(y)\}$. Hence, $\overline{\mu}$ is an AF subalgebra of X.

(4) Let $\overline{\mu}$ be an AF subalgebra of X and $t \in [0,1]$ be such that $U^+(\mu,t) \neq \emptyset$. Let $x,y \in X$ be such that $x,y \in U^+(\mu,t)$. Then $\mu(x) > t$ and $\mu(y) > t$. Thus, $\overline{\mu}(x \cdot y) \leq \max\{\overline{\mu}(x),\overline{\mu}(y)\}$. Then $1-\mu(x \cdot y) \leq \max\{1-\mu(x),1-\mu(y)\} = 1-\min\{\mu(x),\mu(y)\}$. Thus, $\mu(x \cdot y) \geq \min\{\mu(x),\mu(y)\} > t$, so $x \cdot y \in U^+(\mu,t)$. Hence, $U^+(\mu,t)$ is a subalgebra of X.

Conversely, assume that every nonempty set $U^+(\mu,t)$ is a subalgebra in X. Let $x,y \in X$ be such that $\overline{\mu}(x \cdot y) > \max\{\overline{\mu}(x),\overline{\mu}(y)\}$. Then $1-\mu(x \cdot y) > \max\{1-\mu(x),1-\mu(y)\} = 1-\min\{\mu(x),\mu(y)\}$. Thus, $\mu(x \cdot y) < \min\{\mu(x),\mu(y)\}$. Now $\mu(x \cdot y) \in [0,1]$, we choose $t = \mu(x \cdot y)$. Thus, $\mu(x) > t$ and $\mu(y) > t$, so $x,y \in U^+(\mu,t) \neq \emptyset$. By assumption, we have $U^+(\mu,t)$ is a subalgebra of X and so $x \cdot y \in U^+(\mu,t)$. Thus, $\mu(x \cdot y) > t = \mu(x \cdot y)$, which is a contradiction. Hence, $\overline{\mu}(x \cdot y) \leq \max\{\overline{\mu}(x),\overline{\mu}(y)\}$ for all $x,y \in X$. Therefore, $\overline{\mu}$ is an AF subalgebra of X. □

The following two theorems can be proved similarly to Theorem 3.3.

Theorem 3.4. *Let* μ *be a fuzzy set in a Hilbert algebra* $X = (X,\cdot,1)$. *Then the following statements hold:*

(1) μ *is an AF ideal of* X *if and only if it satisfies for all* $t \in [0,1]$, $L(\mu,t) \neq \emptyset$ *implies* $L(\mu,t)$ *is an ideal of* X,

(2) μ *is an AF ideal of* X *if and only if it satisfies for all* $t \in [0,1]$, $L^-(\mu,t) \neq \emptyset$ *implies* $L^-(\mu,t)$ *is an ideal of* X,

(3) $\overline{\mu}$ *is an AF ideal of* X *if and only if it satisfies for all* $t \in [0,1]$, $U(\mu,t) \neq \emptyset$ *implies* $U(\mu,t)$ *is an ideal of* X,

(4) $\overline{\mu}$ *is an AF ideal of* X *if and only if it satisfies for all* $t \in [0,1]$, $U^+(\mu,t) \neq \emptyset$ *implies* $U^+(\mu,t)$ *is an ideal of* X.

Theorem 3.5. *Let μ be a fuzzy set in a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold:*

(1) μ is an AF deductive system of X if and only if it satisfies for all $t \in [0,1]$, $L(\mu, t) \neq \emptyset$ implies $L(\mu, t)$ is a deductive system of X,
(2) μ is an AF deductive system of X if and only if it satisfies for all $t \in [0,1]$, $L^-(\mu, t) \neq \emptyset$ implies $L^-(\mu, t)$ is a deductive system of X,
(3) $\overline{\mu}$ is an AF deductive system of X if and only if it satisfies for all $t \in [0,1]$, $U(\mu, t) \neq \emptyset$ implies $U(\mu, t)$ is a deductive system of X,
(4) $\overline{\mu}$ is an AF deductive system of X if and only if it satisfies for all $t \in [0,1]$, $U^+(\mu, t) \neq \emptyset$ implies $U^+(\mu, t)$ is a deductive system of X.

The following three corollaries are a direct result of Theorems 3.3, 3.4, and 3.5, respectively.

Corollary 3.1. *Let μ be a fuzzy set in a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold:*

(1) if μ is an AF subalgebra of X, then for every $t \in \mathrm{Im}(\mu)$, $L(\mu, t)$ is a subalgebra of X,
(2) if $\overline{\mu}$ is an AF subalgebra of X, then for every $t \in \mathrm{Im}(\mu)$, $U(\mu, t)$ is a subalgebra of X.

Corollary 3.2. *Let μ be a fuzzy set in a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold:*

(1) if μ is an AF ideal of X, then for every $t \in \mathrm{Im}(\mu)$, $L(\mu, t)$ is an ideal of X,
(2) if $\overline{\mu}$ is an AF ideal of X, then for every $t \in \mathrm{Im}(\mu)$, $U(\mu, t)$ is an ideal of X.

Corollary 3.3. *Let μ be a fuzzy set in a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold:*

(1) if μ is an AF deductive system of X, then for every $t \in \mathrm{Im}(\mu)$, $L(\mu, t)$ is a deductive system of X,
(2) if $\overline{\mu}$ is an AF deductive system of X, then for every $t \in \mathrm{Im}(\mu)$, $U(\mu, t)$ is a deductive system of X.

Corollary 3.4. *Let I be a subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold:*

(1) for any $k \in (0, 1]$, there exists an AF subalgebra μ of X such that $L(\mu, t) = I$ for all $t < k$ and $L(\mu, t) = X$ for all $t \geq k$,
(2) for any $k \in [0, 1)$, there exists an AF subalgebra μ of X such that $U(\overline{\mu}, t) = I$ for all $t > k$ and $U(\overline{\mu}, t) = X$ for all $t \leq k$.

Proof: (1) Define a fuzzy set $\mu : X \to [0, 1]$ by

$$(\forall x \in X)\left(\mu(x) = \left\{ \begin{array}{ll} 0 & \text{if } x \in I, \\ k & \text{otherwise} \end{array} \right.\right).$$

Then $L(\mu, t) = I$ for all $t < k$ and $L(\mu, t) = X$ for all $t \geq k$. It follows from Theorem 3.3 (1) that μ is an AF subalgebra of X.

(2) Define a fuzzy set $\gamma : X \to [0, 1]$ by

$$(\forall x \in X)\left(\gamma(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in I, \\ k & \text{otherwise} \end{array} \right.\right).$$

Then $U(\gamma, t) = I$ for all $t > k$ and $U(\gamma, t) = X$ for all $t \leq k$. It follows from Theorem 3.3 (3) that $\overline{\gamma}$ is an AF subalgebra of X. Put $\mu = \overline{\gamma}$. Then μ is an AF subalgebra of X such that $U(\overline{\mu}, t) = I$ for all $t > k$ and $U(\overline{\mu}, t) = X$ for all $t \leq k$. □

The following two corollaries can be proved similarly to Corollary 3.4.

Corollary 3.5. *Let I be an ideal of a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold:*

(1) for any $k \in (0, 1]$, there exists an AF ideal μ of X such that $L(\mu, t) = I$ for all $t < k$ and $L(\mu, t) = X$ for all $t \geq k$,

(2) for any $k \in [0, 1)$, there exists an AF ideal μ of X such that $U(\overline{\mu}, t) = I$ for all $t > k$ and $U(\overline{\mu}, t) = X$ for all $t \leq k$.

Corollary 3.6. *Let I be a deductive system of a Hilbert algebra $X = (X, \cdot, 1)$. Then the following statements hold:*

(1) for any $k \in (0, 1]$, there exists an AF deductive system μ of X such that $L(\mu, t) = I$ for all $t < k$ and $L(\mu, t) = X$ for all $t \geq k$,

(2) for any $k \in [0, 1)$, there exists an AF deductive system μ of X such that $U(\overline{\mu}, t) = I$ for all $t > k$ and $U(\overline{\mu}, t) = X$ for all $t \leq k$.

Definition 3.5. *Let μ_X and μ_Y be fuzzy sets in Hilbert algebras X and Y, respectively. The Cartesian product $\mu_X \times \mu_Y : X \times Y \to [0, 1]$ is defined by*

$$(\forall (x, y) \in X \times Y)((\mu_X \times \mu_Y)(x, y) = \max\{\mu_X(x), \mu_Y(y)\}).$$

Remark 3.1. *Let $X = (X, \cdot, 1_X)$ and $Y = (Y, *, 1_Y)$ be Hilbert algebras. We define the binary operation $\otimes$ on $X \times Y$ by*

$$(\forall (x, y), (u, v) \in X \times Y)((x, y) \otimes (u, v) = (x \cdot u, y * v)).$$

Then $(X \times Y, \otimes, (1_X, 1_Y))$ is a Hilbert algebra.

Proposition 3.7. *If μ_X and μ_Y are AF subalgebras of Hilbert algebras $X = (X, \cdot, 1_X)$ and $Y = (Y, *, 1_Y)$, respectively, then the Cartesian product $\mu_X \times \mu_Y$ is also an AF subalgebra of $X \times Y$.*

Proof: Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}
(\mu_X \times \mu_Y)((x_1, y_1) \otimes (x_2, y_2)) &= (\mu_X \times \mu_Y)(x_1 \cdot x_2, y_1 * y_2) \\
&= \max\{\mu_X(x_1 \cdot x_2), \mu_Y(y_1 * y_2)\} \\
&\leq \max\{\max\{\mu_X(x_1), \mu_X(x_2)\}, \max\{\mu_Y(y_1), \mu_Y(y_2)\}\} \\
&= \max\{\max\{\mu_X(x_1), \mu_Y(y_1)\}, \max\{\mu_X(x_2), \mu_Y(y_2)\}\} \\
&= \max\{(\mu_X \times \mu_Y)(x_1, y_1), (\mu_X \times \mu_Y)(x_2, y_2)\}.
\end{aligned}$$

Hence, $\mu_X \times \mu_Y$ is an AF subalgebra of $X \times Y$. □

The following two propositions can be proved similarly to Proposition 3.7.

Proposition 3.8. *If μ_X and μ_Y are AF ideals of Hilbert algebras $X = (X, \cdot, 1_X)$ and $Y = (Y, *, 1_Y)$, respectively, then the Cartesian product $\mu_X \times \mu_Y$ is also an AF ideal of $X \times Y$.*

Proposition 3.9. *If μ_X and μ_Y are AF deductive systems of Hilbert algebras $X = (X, \cdot, 1_X)$ and $Y = (Y, *, 1_Y)$, respectively, then the Cartesian product $\mu_X \times \mu_Y$ is also an AF deductive system of $X \times Y$.*

4. **Conclusion.** In the present paper, we have introduced the concepts of AF subalgebras, AF ideals, and AF deductive systems of Hilbert algebras. The relationship between AF subalgebras (resp., AF ideals, AF deductive systems) and their level subsets is described. AF subalgebras, AF ideals, and AF deductive systems are also used in the Cartesian product of Hilbert algebras. In addition, the idea of the Cartesian product of fuzzy sets has been introduced.

In the future, we will research intuitionistic fuzzy sets in the concept of anti-type in Hilbert algebras to extend the results of this paper. Interested researchers can apply it to other algebraic systems as well and compare the results with this paper.

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