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BI-FUZZY IDEALS OF d-ALGEBRAS

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Abstract. *In a d-algebra, the concepts of bi-fuzzy subalgebras and ideals are introduced. It was investigated the relationship between bi-fuzzy subalgebras (ideals) and their upper level subsets. Finally, the homomorphic properties of bi-fuzzy ideals are examined.*
Keywords: d-algebra, Ideal, Normal ideal, Bi-fuzzy subalgebra, Bi-fuzzy ideal

1. **Introduction.** The algebraic structures of BCK-algebras and BCI-algebras were studied by Iséki and his colleague [1-3]. In 1983, Hu and Li [4] generalized a new class of algebras from BCI-algebras, namely, a BCH-algebra. Next, Bandru and Rafi [5] introduced a new algebra, called G-algebra. However, in 2012, G-algebra played an important role and many applications. In 2002, Neggers and Kim [6] combined some properties from two algebra, i.e., BCI-algebra and BCH-algebra, and obtained a new algebra, namely, B-algebra. Neggers and Kim [7] introduced the concept of d-algebras in 1999, which is another useful generalization of BCK-algebras, and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs which got the attention of the author as follows [8].

In 2005, Akram and Dar [9] introduced the concepts of fuzzy subalgebras and ideals in d-algebras, and investigated some of their results. In 2010, Muthuraj et al. [10] studied Q-fuzzy BG-ideal of a BG-algebra. They gave condition for Q-fuzzy subsets to be Q-fuzzy BG-ideals. In 2018, Khalil [11] introduced a new category of fuzzy d-algebra. There is

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a relation between fuzzy d-algebras and edge d-algebras. In 2015, Dymek and Walendziak [12] introduced normal ideal and relationship of fuzzy ideals and ideals of BN-algebras. The concept of fuzzy ideals is continually studied in d-algebras, which has inspired us to expand our study to bi-fuzzy ideals by referring to [11,13-19].

We investigate a normal ideal and a congruence of a d-algebra in this paper, and also provide the concept of a bi-fuzzy subalgebra and ideal of a d-algebra. The relationship between bi-fuzzy subalgebras (ideals) and their upper level subsets is given. Finally, the bi-fuzzy ideal's homomorphic properties are discovered.

2. Preliminaries.

We will review the concepts, theorems, and understanding needed to explore the main sections in this subject.

Definition 2.1. [7] *A d-algebra $X = (X, *, 0)$ is a nonempty set X with an element 0 and a binary operation $*$ satisfying the following axioms:*

(d1) $(\forall x \in X)(x * x = 0)$,
(d2) $(\forall x \in X)(0 * x = 0)$,
(d3) $(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y)$.

On a d-algebra $X = (X, *, 0)$, the binary relation $\leq$ is defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0).$$

Example 2.1. *Let $X = \{0, a, b, c\}$ with the following Cayley table as follows:*

$*$	0	a	b	c
0	0	0	0	0
a	c	0	a	a
b	b	c	0	0
c	c	b	a	0

Then $X = (X, *, 0)$ is a d-algebra.

After this we will use X instead of a d-algebra $(X, *, 0)$.

Definition 2.2. [8] *A nonempty subset S of X is called*

(1) *a subalgebra of X if* $(\forall x, y \in S)(x * y \in S)$,
(2) *an ideal of X if*
  (I1) $(\forall x, y \in X)(x * y \in S, y \in S \Rightarrow x \in S)$,
  (I2) $(\forall x \in S, \forall y \in X)(x * y \in S)$.

It is easy to check that $\{0\}$ and X are ideals of X.

We know that if I is an ideal of X, then $0 \in I$, and every ideal of X is a subalgebra.

Definition 2.3. *A nonempty subset N of X is said to be normal in X if*

$$(\forall x, y, a, b \in X)(x * y, a * b \in N \Rightarrow (x * a) * (y * b) \in N).$$

An ideal I of X is called a normal ideal of X if I is normal. In addition, $\mathcal{N}_{id}(X)$ denotes the set of all normal ideals of X.

We know that $X \in \mathcal{N}_{id}(X)$ but $\{0\} \notin \mathcal{N}_{id}(X)$ because $b * c$, $0 * b \in \{0\}$ but $(b * 0) * (c * b) = b * a = c \notin \{0\}$ (see Example 2.1), and every normal ideal of X is a subalgebra.

Proposition 2.1. *If $I \in \mathcal{N}_{id}(X)$, then*

(NI1) $(\forall x, y \in X)(x * y \in I \Rightarrow (x * 0) * (y * 0) \in I)$,
(NI2) $(\forall x, y \in X)((x * 0) * (x * y) \in I)$,
(NI3) $(\forall x, y \in X)(x * y \in I \Leftrightarrow x * 0 \in I)$.

Proof: (NI1) Suppose that $x * y \in I$. Since $0 * 0 = 0 \in I$ (by (d2)), we have $(x * 0) * (y * 0) \in I$.

(NI2) By (d1) and (d2), we have $x * x = 0, 0 * y = 0 \in I$. Thus, $(x * 0) * (x * y) \in I$.

(NI3) The proof of sufficient condition is straightforward by (NI2) and (I1).

Conversely, let $x, y \in X$ be such that $x * 0 \in I$. Since $y * y = 0 \in I$ (by (d1)) and by (d2), we obtain $(x*y)*0 = (x*y)*(0*y) \in I$. Since $0 \in I$ and by (I1), we have $x*y \in I$. □

Definition 2.4. [7] *A d-algebra X is said to be*

(1) *edge if* $(\forall x \in X)(x * X = \{x, 0\})$,

(2) *skew-edge if* $(\forall x \in X)(x * 0 = x)$.

It is known that if X is edge, then it is skew-edge.

Example 2.2. *Let* $X = \{0, a, b, c\}$ *with the following Cayley table as follows:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	b	a
b	b	c	0	a
c	c	0	c	0

Then X is a skew-edge d-algebra but it is not edge because $a * X = \{0, a, b\} \neq \{0, a\}$.

Proposition 2.2. *Let X be a skew-edge d-algebra and S be a nonempty subset of X. Then S is a normal subalgebra of X if and only if* $S \in \mathcal{N}_{id}(X)$.

Proof: The proof of sufficient condition is obvious.

Conversely, suppose that S is a normal subalgebra of X.

(I1) Suppose that $x * y \in S$ and $y \in S$. By (d2), we have $0 * y = 0 \in S$. By (d1), we have $x = (x * 0) * 0 = (x * 0) * (y * y) \in S$.

(I2) Suppose that $x \in S$ and $y \in X$. Then $x * 0 = x \in S$ and $y * y = 0 \in S$. By (d2), we have $x * y = (x * y) * 0 = (x * y) * (0 * y) \in S$.

Hence, $S \in \mathcal{N}_{id}(X)$. □

Let X be a skew-edge d-algebra and $I \in \mathcal{N}_{id}(X)$. We define a binary relation $\smile_I$ on X as follows:

$$(\forall x, y \in X)(x \smile_I y \Leftrightarrow x * y \in I).$$

(reflexivity) Let $x \in X$. By (d1), we have $x * x = 0 \in I$. Thus, $x \smile_I x$.

(symmetry) Let $x, y \in X$ be such that $x \smile_I y$. Then $x * y \in I$. Since $y * y = 0 \in I$ (by (d1)) and by (d1), we have $y * x = (y * x) * 0 = (y * x) * (y * y) \in I$. Thus, $y \smile_I x$.

(transitivity) Let $x, y, z \in X$ be such that $x \smile_I y$ and $y \smile_I z$. By symmetry, we have $z \smile_I y$. Thus, $x * y, z * y \in I$. By (d1), we have $x * z = (x * z) * 0 = (x * z) * (y * y) \in I$. Thus, $x \smile_I z$.

(compatible) Let $x, y, z \in X$ be such that $x \smile_I y$. Then $x * y \in I$. Since $z * z = 0 \in I$ (by (d1)), we have $(x * z) * (y * z) \in I$ and $(z * x) * (z * y) \in I$. Thus, $x * z \smile_I y * z$ and $z * x \smile_I z * y$.

Therefore, $\smile_I$ is a congruence on a skew-edge d-algebra X.

Denote the equivalence class containing X by $[x]_I$, i.e., $[x]_I = \{y \in X \mid x \smile_I y\}$ and let $X/I = \{[x]_I \mid x \in X\}$.

We define a binary operation $\star$ on X/I as follows:

$$(\forall x, y \in X)([x]_I \star [y]_I = [x * y]_I).$$

The following theorem is obtained.

Theorem 2.1. *Let X be a skew-edge d-algebra and* $I \in \mathcal{N}_{id}(X)$. *Then* $(X/I, \star, [0]_I)$ *is also a skew-edge d-algebra.*

Proof: (d1) Let $[x]_I \in X/I$. Then $[x]_I \star [x]_I = [x * x]_I = [0]_I$.

(d2) Let $[x]_I \in X/I$. Then $[0]_I \star [x]_I = [0 * x]_I = [0]_I$.

(d3) Let $[x]_I, [y]_I \in X/I$ be such that $[x]_I \star [y]_I = [0]_I$ and $[y]_I \star [x]_I = [0]_I$. Then $[x * y]_I = [y * x]_I = [0]_I$. Thus, $x * y \smile_I 0$, so $x * y = (x * y) * 0 \in I$. Thus, $x \smile_I y$, so $[x]_I = [y]_I$.

(skew-edge) Let $[x]_I \in X/I$. Then $[x]_I \star [0]_I = [x * 0]_I = [x]_I$.

Hence, $(X/I, \star, [0]_I)$ is a skew-edge d-algebra and it is called a *quotient d-algebra.* □

Definition 2.5. [21] *A d-algebra* $X = (X, *, 0)$ *is said to be medial if* $(\forall x, y, z \in X)((x * y) * (z * u) = (x * z) * (y * u))$.

The binary operation $\sqcap$ on X is defined by

$$(\forall x, y \in X)(x \sqcap y = (y * x) * x).$$

3. **Main Results.** In this section, we introduce the concepts of bi-fuzzy subalgebras and ideals of d-algebras and study the relationship between bi-fuzzy subalgebras (ideals) and subalgebras (ideals).

3.1. Bi-fuzzy subalgebras and ideals.

Definition 3.1. *A bi-fuzzy set* δ *of a nonempty set* A *is a mapping* $\delta : A \times A \to [0, 1]$. *In particular, a fuzzy set* ν *of a nonempty set* A *is a mapping* $\nu : A \to [0, 1]$.

Definition 3.2. *Let* δ *be a bi-fuzzy set of a nonempty set* A. *For* $t \in [0, 1]$, *the set* $\delta_t = \{(x, y) \in A \times A \mid \delta(x, y) \geq t\}$ *is called an upper level subset of* δ.

Definition 3.3. *A bi-fuzzy set* δ *of* X *is called a bi-fuzzy subalgebra of* X *if it satisfies*

$$(\forall (x, u), (y, v) \in X \times X)(\delta(x * y, u * v) \geq \min\{\delta(x, u), \delta(y, v)\}).$$

Example 3.1. *In Example 2.1, we define a bi-fuzzy set* δ *of* X *by*

$$(\forall (x, y) \in X \times X)\left(\delta(x, y) = \begin{cases} 0.58 & \text{if } x = y = 0, \\ 0 & \text{otherwise.} \end{cases}\right)$$

Then δ *is a bi-fuzzy subalgebra of* X. *In addition,* $\delta_{0.58} = \{(0, 0)\}$ *and* $\delta_0 = X \times X$.

For a d-algebra X, we define a binary operation $\circledast$ on $X \times X$ by

$$(\forall x, y, u, v \in X)((x, u) \circledast (y, v) = (x * y, u * v)).$$

(d1) Let $(x, y) \in X \times X$. Then $(x, y) \circledast (x, y) = (x * x, y * y) = (0, 0)$.

(d2) Let $(x, y) \in X \times X$. Then $(0, 0) \circledast (x, y) = (0 * x, 0 * y) = (0, 0)$.

(d3) Let $(x, y), (u, v) \in X \times X$ be such that $(x, y) \circledast (u, v) = (0, 0)$ and $(u, v) \circledast (x, y) = (0, 0)$. Then $x * u = 0 = u * x$ and $y * v = 0 = v * y$. This means that $(x, y) = (u, v)$.

Hence, $(X \times X, \circledast, (0, 0))$ is a d-algebra.

Proposition 3.1. *A bi-fuzzy set* δ *of* X *is a bi-fuzzy subalgebra if and only if for every* $t \in [0, 1]$, *the upper level subset* δ_t *is either empty or a subalgebra of* $X \times X$.

Proof: Suppose that δ is a bi-fuzzy subalgebra of X. Let $t \in [0, 1]$ be such that $\delta_t \neq \emptyset$. Then $\delta(x * y, u * v) \geq \min\{\delta(x, u), \delta(y, v)\} \geq t$ for all $(x, u), (y, v) \in \delta_t$. This implies that $(x, u) \circledast (y, v) = (x * y, u * v) \in \delta_t$ for all $(x, u), (y, v) \in \delta_t$. Hence, δ_t is a subalgebra of $X \times X$.

Conversely, suppose that for every $t \in [0, 1]$, the upper level subset δ_t is either empty or a subalgebra of $X \times X$. Let $(x, u), (y, v) \in X \times X$. Choose $t = \min\{\delta(x, u), \delta(y, v)\}$. Then $(x, u), (y, v) \in \delta_t \neq \emptyset$. By assumption, δ_t is a subalgebra of X. This implies that $(x * y, u * v) = (x, u) \circledast (y, v) \in \delta_t$. Thus, $\delta(x * y, u * v) \geq t = \min\{\delta(x, u), \delta(y, v)\}$. Hence, δ is a bi-fuzzy subalgebra of X. □

Theorem 3.1. *Any subalgebra of a d-algebra $X \times X$ can be (realized as) a level subalgebra of some bi-fuzzy subalgebra of X.*

Proof: Let S be a subalgebra of a d-algebra $X \times X$. We define a bi-fuzzy set δ of X by

$$(\forall (x,y) \in X \times X) \left(\delta(x,y) = \begin{cases} c & \text{if } (x,y) \in S, \\ 0 & \text{otherwise,} \end{cases} \right)$$

where $c \in (0,1)$. Then $\delta_c = S$. Let $(x,u),(y,v) \in X \times X$.

Case 1: $(x,u),(y,v) \in S$. Then $(x*y, u*v) = (x,u) \circledast (y,v) \in S$. This implies that $\delta(x,u) = \delta(y,v) = \delta(x*y,u*v) = c$. Thus, $\delta(x*y,u*v) \geq \min\{\delta(x,u),\delta(y,v)\}$.

Case 2: $(x,u) \notin S$ or $(y,v) \notin S$. Then $\delta(x,u) = 0$ or $\delta(y,v) = 0$. This implies that $\delta(x*y,u*v) \geq 0 = \min\{\delta(x,u),\delta(y,v)\}$.

Hence, δ is a bi-fuzzy subalgebra of X. □

Definition 3.4. *For any bi-fuzzy sets δ and γ in a nonempty set A, we define a binary relation $\leq$ as follows:*

$$\delta \leq \gamma \Leftrightarrow \delta(x,y) \leq \gamma(x,y) \quad \forall (x,y) \in A \times A.$$

Let A and B be nonempty sets, a function $f : A \times A \to B$, and a bi-fuzzy set δ of A. Set $f^{\leftarrow}(z) = \{(x,y) \in A \times A \mid f(x,y) = z\}$ for $z \in B$. The fuzzy set γ of B is defined by

$$(\forall z \in B) \left(\gamma(z) = \begin{cases} \sup\{\delta(x,y) \mid (x,y) \in f^{\leftarrow}(z)\} & \text{if } f^{\leftarrow}(z) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \right)$$

Then γ is called the *image* of δ under f and is denoted by $f(\delta)$.

Let A and B be nonempty sets, a function $f : A \times A \to B$, and a fuzzy set γ of $f(A \times A)$. The bi-fuzzy set δ of $A \times A$ is defined by

$$(\forall (x,y) \in A \times A)(\delta(x,y) = \gamma(f(x,y))).$$

Then δ is called the *preimage* of γ under f and is denoted by $f^{\leftarrow}(\gamma)$.

Now, we give the concept of a bi-fuzzy ideal in a d-algebra.

Definition 3.5. *A bi-fuzzy set δ of X is called a bi-fuzzy ideal of X if*

(fd1) $(\forall (x,y) \in X \times X)(\delta(0,0) \geq \delta(x,y))$,
(fd2) $(\forall (x,y),(u,v) \in X \times X)(\delta(x,y) \geq \min\{\delta((x,y) \circledast (u,v)),\delta(u,v)\})$.

Definition 3.6. *A fuzzy set γ of X is called a fuzzy ideal of X if*

(fi1) $(\forall x \in X)(\gamma(0) \geq \gamma(x))$,
(fi2) $(\forall x \in X)(\gamma(x) \geq \min\{\gamma(x*y),\gamma(y)\})$.

Example 3.2. *From Example 3.1, we have δ is a bi-fuzzy ideal of X.*

Proposition 3.2. *Let δ be a bi-fuzzy ideal of X. Then*

$$(\forall (x,y),(u,v) \in X \times X)((x,y) \leq (u,v) \Rightarrow \delta(x,y) \geq \delta(u,v)).$$

Proof: Let $(x,y),(u,v) \in X \times X$ be such that $(x,y) \leq (u,v)$. Then $(0,0) = (x,y) \circledast (u,v) = (x*u,y*v)$. Thus, $\delta(x,y) \geq \min\{\delta((x,y) \circledast (u,v)),\delta(u,v)\} = \min\{\delta(0,0),\delta(u,v)\} = \delta(u,v)$. □

Denote by $\mathcal{BF}_{id}(X)$ the set of all bi-fuzzy ideals of X.

Proposition 3.3. *A bi-fuzzy set δ of X is a bi-fuzzy ideal of X if and only if it satisfies*

(fd1) $(\forall (x,y) \in X \times X)(\delta(0,0) \geq \delta(x,y))$,
(fd3) $(\forall (x,y),(u,v),(w,z) \in X \times X)(((x,y) \circledast (u,v)) \circledast (w,z) = (0,0) \Rightarrow \delta(x,y) \geq \min\{\delta(w,z),\delta(u,v)\})$.

Proof: Suppose that δ is a bi-fuzzy ideal of X. Then it satisfies (fd1). Let $(x,y),(u,v),(w,z) \in X \times X$ be such that $((x,y) \circledast (u,v)) \circledast (w,z) = (0,0)$. Using (fd2), we have $\delta(x*u, y*v) \geq \min\{\delta((x*u)*w, (y*v)*z), \delta(w,z)\} = \min\{\delta(0,0), \delta(w,z)\} = \delta(w,z)$ and $\delta(x,y) \geq \min\{\delta(x*u, y*v), \delta(u,v)\}$. This implies that $\delta(x,y) \geq \min\{\delta(w,z), \delta(u,v)\}$.

Conversely, let $(x,y),(u,v) \in X \times X$. Note that $((x,y) \circledast (u,v)) \circledast (x*u, y*v) = (0,0)$. By (fd3), we have $\delta(x,y) \geq \min\{\delta(x*u, y*v), \delta(u,v)\} = \min\{\delta((x,y) \circledast (u,v)), \delta(u,v)\}$. Hence, δ is a bi-fuzzy ideal of X. □

Theorem 3.2. *Let δ be a bi-fuzzy set of X. Assume that δ_t satisfies (I2) for all $t \in [0,1]$. Then δ is a bi-fuzzy ideal of X and if and only if for any $t \in [0,1]$, δ_t is an ideal of $X \times X$ if δ_t is nonempty.*

Proof: Suppose that δ is a bi-fuzzy ideal of X and δ_t satisfies (I2). Let $t \in [0,1]$ be such that $\delta_t \neq \emptyset$.

(I1) Assume that $(x,y) \circledast (u,v) \in \delta_t$ and $(u,v) \in \delta_t$. Then $\delta((x,y) \circledast (u,v)) \geq t$ and $\delta(u,v) \geq t$. By (fd2), we have $\delta(x,y) \geq \min\{\delta((x,y) \circledast (u,v)), \delta(u,v)\} \geq t$. Thus, $(x,y) \in \delta_t$.

Therefore, δ_t is an ideal of $X \times X$.

Conversely, suppose that δ_t is an ideal of $X \times X$ for any $t \in [0,1]$ and δ_t is nonempty.

(fd1) Assume that there exists $(x,y) \in X \times X$ such that $\delta(0,0) < \delta(x,y) = c$ for some $c \in [0,1]$. Then $(x,y) \in U(\delta,c) \neq \emptyset$. By assumption, we have $U(\delta,c)$ is an ideal of $X \times X$. This means that $(0,0) \in U(\delta,c)$, that is, $\delta(0,0) \geq c$. It is a contradiction. Thus, for each $(x,y) \in X \times X$, $\delta(0,0) \geq \delta(x,y)$.

(fd2) Assume that there exist $(x,y),(u,v) \in X \times X$ such that $\delta(x,y) < \min\{\delta(x*u, y*v), \delta(u,v)\}$. Choosing $c = \frac{1}{2}(\delta(x,y) + \min\{\delta(x*u, y*v), \delta(u,v)\})$, we get $\delta(x,y) < \frac{1}{2}(\delta(x,y) + \min\{\delta(x*u, y*v), \delta(u,v)\}) = c < \min\{\delta(x*u, y*v), \delta(u,v)\} \leq \delta(x*u, y*v)$ and $c < \delta(u,v)$. Since $(x*u, y*v), (u,v) \in U(\delta,c)$ and $U(\delta,c)$ is an ideal of $X \times X$, we have $(x,y) \in U(\delta,c)$, that is, $\delta(x,y) \geq c$. It is a contradiction.

Hence, δ is a bi-fuzzy ideal of X. □

Proposition 3.4. *If δ is a bi-fuzzy ideal of a medial d-algebra X, then*

$$X_\delta = \{(x,y) \in X \times X \mid \delta(x,y) = \delta(0,0)\}$$

is an ideal of $X \times X$.

Proof: Assume that δ is a bi-fuzzy ideal of a medial d-algebra X. Let $(x,y) \circledast (u,v) \in X_\delta$ and $(u,v) \in X_\delta$. Then $\delta(x*u, y*v) = \delta(0,0)$ and $\delta(u,v) = \delta(0,0)$. By (fd1), we have $\delta(0,0) \geq \delta(x,y)$. Using (fd2), we get $\delta(x,y) \geq \min\{\delta(x*u, y*v), \delta(u,v)\} = \min\{\delta(0,0), \delta(0,0)\} = \delta(0,0)$. Thus, $\delta(x,y) = \delta(0,0)$, that is, $(x,y) \in X_\delta$. Next, let $(x,y) \in X_\delta$ and $(u,v) \in X \times X$. By (fd1), we have $\delta(0,0) \geq \delta(x*u, y*v)$. Using (fd2), we get

$$\begin{aligned}
\delta(x*u, y*v) &\geq \min\{\delta((x*u)*0, (y*v)*0), \delta(0,0)\} \\
&= \min\{\delta((x*u)*(x*x), (y*v)*(y*y)), \delta(0,0)\} && \text{(d1)} \\
&= \min\{\delta((x*x)*(u*x), (y*y)*(v*y)), \delta(0,0)\} && \text{(medial)} \\
&= \min\{\delta(0,0), \delta(0,0)\} && \text{(d1, d2)} \\
&= \delta(0,0).
\end{aligned}$$

Thus, $\delta(x*u, y*v) = \delta(0,0)$, that is, $(x,y) \circledast (u,v) \in X_\delta$. Hence, X_δ is an ideal of $X \times X$. □

Lemma 3.1. *Let $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ be a strictly ascending sequence of ideals of $X \times X$ and (c_n) be a strictly decreasing sequence in $(0, 1)$. Define a bi-fuzzy set δ of X by*

$$\delta(x,y) = \begin{cases} 0, & \text{if } (x,y) \notin I_n \text{ for any } n \in \mathbb{N}, \\ c_n, & \text{if } (x,y) \in I_n \text{ for the least } n \in \mathbb{N}. \end{cases}$$

Then δ is a bi-fuzzy ideal of X.

Proof: Let $I = \cup_{n\in\mathbb{N}} I_n$. Then I is an ideal of $X \times X$. By the definition of δ, we get $\delta(0,0) = c_1 \geq \delta(x,y)$ for all $(x,y) \in X \times X$, i.e., (fd1) holds. Let $(x,y),(u,v) \in X \times X$. We separate it into 2 cases.

If $(x,y) \notin I$, then $(x * u, y * v) \notin I$ or $(u,v) \notin I$. Thus, $\delta(x,y) = 0 = \min\{\delta(x * u, y * v), \delta(u,v)\}$.

If $(x,y) \in I_n$ for the least $n \in \mathbb{N}$, then $(x * u, y * v) \notin I_{n-1}$ or $(u,v) \notin I_{n-1}$. That is, $\delta(x * u, y * v) \leq c_n$ or $\delta(u,v) \leq c_n$. Thus, $\delta(x,y) = c_n \geq \min\{\delta(x * u, y * v), \delta(u,v)\}$. Hence, δ is a bi-fuzzy ideal of X. □

3.2. **Homomorphic properties of bi-fuzzy ideals.** Let $(A, *_A, 0_A)$ and $(B, *_B, 0_B)$ be d-algebras. A mapping $f : A \to B$ is called a *homomorphism* from A into B if $f(x *_A y) = f(x) *_B f(y)$ for all $x, y \in A$. The following results give the homomorphic properties of bi-fuzzy ideals.

Theorem 3.3. *Let X and Y be d-algebras, $f : X \times X \to Y$ a homomorphism and γ a fuzzy ideal of Y. Then $f^{\leftarrow}(\gamma)$ is a bi-fuzzy ideal of X.*

Proof: Let $(x,y) \in X \times X$. Then $f(x,y) \in Y$. Since γ is a fuzzy ideal of Y, we have $(f^{\leftarrow}(\gamma))(0,0) = \gamma(f(0,0)) = \gamma(0) \geq \gamma(f(x,y)) = (f^{\leftarrow}(\gamma))(x,y)$. Thus, $f^{\leftarrow}(\gamma)$ satisfies (fd1). Next, let $(x,y),(u,v) \in X \times X$. Since γ is a fuzzy ideal of Y, we have $\gamma(f(x,y)) \geq \min\{\gamma(f(x,y) *_Y f(u,v)), \gamma(f(u,v))\} = \min\{\gamma(f((x,y) \circledast (u,v))), \gamma(f(u,v))\}$. This means that $(f^{\leftarrow}(\gamma))(x,y) \geq \min\{(f^{\leftarrow}(\gamma))((x,y) \circledast (u,v)), (f^{\leftarrow}(\gamma))(u,v)\}$. Hence, $f^{\leftarrow}(\gamma)$ is a bi-fuzzy ideal of X. □

Lemma 3.2. *Let X and Y be d-algebras, $f : X \times X \to Y$ a homomorphism and δ a bi-fuzzy ideal of X. If δ is constant on $\ker(f)$, then $f^{\leftarrow}(f(\delta)) = \delta$.*

Proof: Suppose that δ is constant on $\ker(f) = f^{\leftarrow}(0)$. Let $(x,y) \in X \times X$. Then there exists $z \in Y$ such that $f(x,y) = z$. Thus,

$$(f^{\leftarrow}(f(\delta)))(x,y) = (f(\delta))(f(x,y)) = (f(\delta))(z) = \sup\{\delta(u,v) \mid (u,v) \in f^{\leftarrow}(z)\}.$$

Let $(u,v) \in f^{\leftarrow}(z)$. Then $f(x,y) = f(u,v)$. This implies that $f((u,v) \circledast (x,y)) = 0_Y$, i.e., $(u,v) \circledast (x,y) \in \ker(f)$. Thus, $\delta((u,v) \circledast (x,y)) = \delta(0,0)$. Therefore,

$$\delta(u,v) \geq \min\{\delta(u * x, v * y), \delta(x,y)\} = \min\{\delta(0,0), \delta(x,y)\} = \delta(x,y).$$

Similarly, we get $\delta(x,y) \geq \delta(u,v)$. Hence, $\delta(x,y) = \delta(u,v)$. Thus,

$$(f^{\leftarrow}(f(\delta)))(x,y) = \sup\{\delta(u,v) \mid (u,v) \in f^{\leftarrow}(z)\} = \delta(x,y).$$

Hence, $f^{\leftarrow}(f(\delta)) = \delta$. □

Theorem 3.4. *Let X and Y be d-algebras, $f : X \times X \to Y$ a subjective homomorphism and δ a bi-fuzzy ideal of X such that $X_\delta \supseteq \ker(f)$. Then $f(\delta)$ is a fuzzy ideal of Y.*

Proof: Since δ is a bi-fuzzy ideal of X and $(0,0) \in f^{\leftarrow}(0_Y)$, we have

$$(f(\delta))(0_Y) = \sup\{\delta(u,v) \mid (u,v) \in f^{\leftarrow}(0_Y)\} = \delta(0,0) \geq \delta(x,y)$$

for all $(x,y) \in X \times X$. Thus,

$$(f(\delta))(0_Y) = \sup\{\delta(x,y) \mid (x,y) \in f^{\leftarrow}(z)\} = (f(\delta))(z)$$

for all $z \in Y$, that is, $f(\delta)$ satisfies (fi1). Next, suppose that there exist $z, w \in Y$ such that $f(\delta)(z) < \min\{f(\delta)(z *_Y w), f(\delta)(w)\}$. Since f is subjective, there exist $(x, y), (u, v) \in X \times X$ such that $f(x, y) = z$ and $f(u, v) = w$. Thus, $(f(\delta))(f(x, y)) < \min\{(f(\delta))(f(x * u, y * v)), (f(\delta))(f(u, v))\}$. This implies that $(f^{\leftarrow}(f(\delta)))(x, y) < \min\{(f^{\leftarrow}(f(\delta)))(x*u, y* v), (f^{\leftarrow}(f(\delta)))(u, v)\}$. Since $X_\delta \supseteq \ker(f)$, we have δ is constant on $\ker(f)$. By Lemma 3.2, we have $\delta(x, y) < \min\{\delta(x * u, y * v), \delta(u, v)\}$. It is a contradiction to the fact that δ is a bi-fuzzy ideal of X. Thus, $f(\delta)$ satisfies (fi2). Hence, $f(\delta)$ is a fuzzy ideal of Y. □

4. **Conclusion and Discussion.** In a d-algebra, we have given the properties of a normal ideal. A normal ideal can also be used to generate the quotient skew-edge d-algebra. Following that, a bi-fuzzy subalgebra and ideal of a d-algebra was introduced. It is possible to get their properties. Finally, the homomorphic properties of a bi-fuzzy ideal are also given. A topic of interest and research in algebra, such as BF/BO/BM/BH/BG-algebras, is examining the properties of a normal ideal and a bi-fuzzy ideal.

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