ICIC Express Letters
Part B: Applications
Volume **14**, Number **12**, December **2023**

ICIC International ©2023 ISSN 2185-2766
pp. 1267–1272

IDENTIFICATION OF NONLINEAR TIME-VARIANT SYSTEMS BASED ON NEURAL NETWORKS

Hongpeng Li[1,2], Sicheng Zhang[1] and Aochun Wang[1]

[1]College of Science
Jiangsu Ocean University
[2]Jiangsu Marine Resources Development Institute
No. 59, Cangwu Road, Haizhou District, Lianyungang 222005, P. R. China
2019000027@jou.edu.cn; rain.siki@foxmail.com; 932586280@qq.com

Received April 2023; accepted June 2023

Abstract. *The main goal of this study is to elucidate the theoretical approximation capability of neural networks with time-variant weight. In this paper, we show that the approximation capability of the time-variant network can be extended to time-variant dynamical systems. Moreover, the algorithm is designed according to the expression form of 2-D continuous-discrete system. Finally, one simulation of time-variant neural networks is given to shown effectiveness of the obtained results.*
Keywords: Approximation, Neural networks, Time-variant, Identification, Dynamical system

1. **Introduction.** In recent years, considerable attention has been paid to the application of neural networks in the modeling and identification of dynamic processes. This focus is mostly due to the increasing demand for system identification [1-4], intelligent control systems [5-7], etc. There are two types of connections in neural networks. Neural networks with only feed-forward connections are called feed forward networks, and neural networks with arbitrary connections are often called recurrent neural networks. Clarifying the theoretical capabilities of these neural networks provides important information for learning algorithms and their applications [8-12]. In terms of the theoretical capability of the recurrent neural networks, the approximation possibility to several dynamical systems has been considered. A given trajectory of a dynamical system can be approximately realized by an appropriate continuous time recurrent network. The similar results of the discrete time recurrent network for discrete systems and continuous time recurrent network for non-autonomous systems have been shown [13-16].

The objective of this manuscript is to study the approximation capability of time-variant neural networks. It is shown that the approximation capability is extended to time-variant dynamical systems. The results presented in this paper confirm the use of time-variant neural networks approach to the time-variant dynamical system. The rest of this paper is organized as follows. Theoretical proofs of approximation capability of time-variant neural networks are presented in Section 2. Algorithm based on time-varying neural network is designed in Section 3. In Section 4, the simulation results are obtained. A brief discussion and summary concerning the main results are provided in Section 5.

2. **Approximation Capability of Time-Variant Neural Networks.** In practical situations, there exist many nonlinear time-variant systems. So in this section, based on the continuous neural networks, the time-variant neural networks that can approximate any nonlinear time-varying systems are studied.

DOI: 10.24507/icicelb.14.12.1267

Consider the neural networks with time variant weight:

$$\frac{\partial z(x,t)}{\partial t} = \alpha z(x,t) + W(t)\sigma(z(x,t)) \tag{1}$$

where $\sigma(\cdot)$ is a C^1-sigmoid nonlinear activation function, $z \in R^L$ and α is a fixed constant, chosen as $0 < \alpha < 1$. $W(t)$ is a weight matrix. We will show that time-variant neural networks in the form of Equation (1) with an arbitrary positive small α are capable of approximating the dynamics of a time-variant nonlinear system. The following lemma is the basis of the approximation of time-variant nonlinear systems using neural networks.

Lemma 2.1. *Let the continuous time-variant mapping relation be $f(x,t) : \Omega \times [0,T] \to R$ and set the neural network approximation accuracy $\varepsilon_{lf} > 0$. For an arbitrary $t_i \in [0,T]$, there exists $f(x,t_i) = f_i(x)$; select the neural network basis function $\sigma_i(x) : \Omega \to R^L$, the weight vector $W_i^* \in R^{l_i}$ and an arbitrary positive small α (α is an $n \times n$ diagonal matrix), which satisfies the approximation accuracy. In this case, for any continuous time-variant weight vector $W^*(t) : [0,T] \to R^{l_i}$ that meets $W^*(t_i) = W_i^*$, there exists $\delta_i > 0$, such that, for all $|t - t_i| < \delta_i$*

$$f(x,t) = -\alpha x(t) + W(t)\sigma_i(x(t)) + \varepsilon(x,t)$$

holds.

Proof: For fixed $t_i \in [0,T]$, there exists a fixed continuous function $f(x,t_i) = f_i(x)$. Hence, there exists an optimal time-variant weight function vector $W_i^* \in R^{l_i}$ and a neural network basis function $\sigma_i(x) : \Omega \to R^{l_i}$ such that

$$f_i(x) = W_i^* \sigma_i(x) + \varepsilon_i(x)$$

where $|\varepsilon_i(x)| < \varepsilon_{lf}$, and l_i is the node number of the corresponding neural network at this moment. Let $f(x,t)$ be a nonlinear function for each $t \in [0,T]$, without considering the global approximation accuracy, for any $W^*(t_i) = W_i^*$, there exists a continuous time-variant weight vector function $W^*(t) : [0,T] \to R^{l_i}$ such that

$$f(x,t) = W^*(t)\sigma_i(x) + \varepsilon(x,t) \quad \forall x \in \Omega,\, t \in [0,T]$$

where $\varepsilon(x,t)$ is the network reconstruction error and $\varepsilon(x,t_i) = \varepsilon_i(x)$.

Let $F(x,t) = -\alpha x(t) + W^*(t)\sigma_i(x(t)) + \varepsilon(x,t)$ for $\forall x \in \Omega$ and $|\alpha(t)| < \varepsilon_\alpha$. Next, we need to consider that within the scope of t, the inequality $\max_{x\in\Omega} |\varepsilon(x,t)| < \varepsilon_{lf}$ holds. Based on the definition of the network reconstruction error, we have

$$\varepsilon(x,t) = F(x,t) + \alpha x - W^*(t)\sigma_i(x) = F(x,t) + \alpha x - f(x,t_i) - W^*(t)\sigma_i(x) \tag{2}$$

Take $f_i(x) = W_i^*(t)\sigma_i(x) + \varepsilon_i(x)$ into (2), and take the absolute values of both sides.

$$|\varepsilon(x,t)| \le |F(x,t) - f_i(x)| + \|W^*(t) - W_i^*(t)\|\|\sigma_i(x)\| + |\varepsilon_i(x)| + |\alpha x| \tag{3}$$

Because $F(x,t)$ is uniformly continuous, by the definition of uniform continuity, for an arbitrary $\varepsilon_f > 0$, there exists $\delta_f > 0$ such that for $\|t - t_i\| < \delta_f$

$$|F(x,t) - f_i(x)| < \varepsilon_f$$

holds. Similarly, because $W^*(t)$ is continuous, for an arbitrary positive $\varepsilon_f > 0$, there exists $\varepsilon_W > 0$ such that for all $\|t - t_f\| < \delta_W$, $\|W^*(t) - W_i^*\| < \varepsilon_W$. In view of the basis function $\sigma_i(x)$ being a Gaussian model and the finite dimension for each element in $[0,1]$, there exists $\sigma_{\max}(x) > 0$ such that $\|\sigma_i(x)\| \le \sigma_{\max}(x)$. Considering all of the elements as mentioned above, take into Equation (3)

$$|\varepsilon(x,t)| \le |F(x,t) - f_i(x)| + \|W^*(t) - W_i^*(t)\|\|\sigma_{\max}(x)\| + \max_{x\in\Omega} |\varepsilon_i(x)| + |\alpha x| \tag{4}$$

Based on the previous analysis, we can adjust the neighboring domain of t_i to make $|F(x,t) - f_i(x)|$ and $\|W^*(t) - W_i^*(t)\|$ sufficiently small, where $\sigma_{\max}(x)$ is a fixed value and $\max_{x\in\Omega} |\varepsilon_i(x)| < \varepsilon_{lf}$. Therefore, there must be a nearby neighborhood of t_i such that

$$|F(x,t) - f_i(x)| + \|W^*(t) - W_i^*(t)\| \|\sigma_{\max}(x)\| + \max_{x\in\Omega} |\varepsilon_i(x)| + |\alpha x| < \varepsilon_{lf} \tag{5}$$

Defining $E = \{(\varepsilon_{lf}, \varepsilon_W, \varepsilon_\alpha) | \varepsilon_{lf} > 0,\ \varepsilon_W > 0,\ \varepsilon_\alpha > 0,\ \varepsilon_{lf} + \varepsilon_W \sigma_{\max} + \varepsilon_\alpha = \varepsilon_{lf} - \max_{x\in\Omega} |\varepsilon_i(x)|\}$, the meaning of E can satisfy the inequality in Equation (4) such that $|f(x,t) - f_i(x)|$ and $\|W^*(t) - W_i^*\|$ are the upper bounds of the collection. Define $\Delta_i = \{\delta | (\varepsilon_f, \varepsilon_W, \varepsilon_\alpha) \in E\}$, for all $\|t - t_i\| < \delta$, $\forall x \in \Omega$, $|f(x,t) - f_i(x)| < \varepsilon_f$, $\|W^*(t) - W_i^*\| < \varepsilon_W$. The meaning of Δ_i is the set of δ neighbors near t_i that satisfy the inequality in Equation (5). Let $\delta_i = \max\{\Delta_i\}$ and then, lemma can be obtained.

The following theorem is the basis for the approximation of the time-variant system using the time-variant neural network.

Theorem 2.1. *Let $\Omega \subset R^n$ be an open set, $D_\Omega \subset \Omega$ be a compact set, and $f(x,t) : \Omega \times [0,T] \to R^n$ be a C^1-class vector function. For a time-variant system,*

$$\dot{x} = f(x(t), t) \quad x \in \Omega \quad t \in [0,T]$$

Set the neural network approximation accuracy ε_{lf} and the fixed constant α to satisfy $0 < \alpha < 1$. Given this, there exists a piecewise continuous optimal time-varying weight vector function $W^*(t) : [0,T] \to R^L$ and a corresponding basis function neural network $\sigma(x) : \Omega \to R^L$ such that

$$f(x(t), t) = -\alpha x(t) + W^*(t)\sigma(x(t)) + \varepsilon(x,t) \quad x \in \Omega \quad t \in [0,T]$$

where $\varepsilon(x,t) = f(x,t) - W^*(t)\sigma(x(t))$ is the approximate error and $\varepsilon(x,t)$ is a piecewise continuous function that satisfies $|\varepsilon(x,t)| \le \varepsilon_{lf}$.

Proof: Lemma 2.1 shows that, for any time $t_i \in [0,T]$, the method can approximate the nonlinear mapping of a neural network, on the premise of guaranteed accuracy of the expanded field near the moment. Due to the limited $[0,T]$, there must be a finite time sequence $\{t_i\}$ and corresponding neural network basis functions $\{\sigma_i(\cdot)\}$, that can maintain the precision extension field sequence $\{N_i\}$ ($N_i = \{t || t - t_i| < \delta_i\}, i = 1, 2, \ldots, n$) covering the whole period $[0,T]$. In every neighborhood, the corresponding neural network basis function can approximate the nonlinear time-variant function, and the precision of the approximation error satisfies the given requirement.

Based on Lemma 2.1, suppose that t_i corresponds to the basis function neural network dimension l_i. Then, $\sigma_i(\cdot) : \Omega \to R^{l_i}$, where $\sigma_i(\cdot) = \left[\sigma_i^1(x), \sigma_i^2(x), \ldots, \sigma_i^{l_i}(x)\right]$, and the weight vector for the continuous time-variant neural network is $W_i^*(t) : [0,T] \to R^{l_i}$, where $W_i^*(t) = \left[W_{i1}^*(t), W_{i2}^*(t), \ldots, W_{il_i}^*(t)\right]^T$. Then, for $|t - t_i| < \delta_i$, there exists

$$F(x,t) = -\alpha x(t) + W_i^{*T}(t)\sigma_i(x(t) + \varepsilon_i(x,t))$$

where $\varepsilon_i(x,t)$ is the network reconstruction error, and satisfies $|\varepsilon_i(x,t)| < \varepsilon_{lf}$.

Considering the time series $\{t_i\}$ that covers $[0,T]$ and their corresponding neighborhood sequence $\{N_i\}$, without loss of generality, assume that the sequence of the selected areas and any two adjacent neighborhood overlap. Then, the function $F(x,t)$ can be expressed as a piecewise function in the following form:

$$F(x,t) = \begin{cases} -\alpha x(t) + W_1^{*T}(t)\sigma_1(x(t) + \varepsilon_1(x,t)) & 0 \le t \le t_1 + \delta_1 \\ -\alpha x(t) + W_2^{*T}(t)\sigma_2(x(t) + \varepsilon_2(x,t)) & t_1 + \delta_1 \le t \le t_2 + \delta_2 \\ & \vdots \\ -\alpha x(t) + W_n^{*T}(t)\sigma_n(x(t) + \varepsilon_n(x,t)) & t_{n-1} + \delta_{n-1} \le t \le T \end{cases}$$

Define $W^*(t) = \left[V_1^{*T}(t), V_2^{*T}(t), \ldots, V_n^{*T}(t)\right]^T$ and $\sigma(x) = \left[\sigma_1^T(x), \sigma_2^T(x), \ldots, \sigma_n^T(x)\right]^T$ where $V_i^*(t) : [0,T] \to R^L$, $i = 1, 2, \ldots, n$, and let

$$V_i^*(t) = \begin{cases} W_i^*(t) & t_i - \delta_i \le t \le t_i + \delta_i \\ [0]_{l_i \times L} & t < t_i - \delta \text{ or } t > t_i + \delta_i \end{cases}$$

Obviously, $W^*(t) : [0,T] \to R^{l_i}$ is a piecewise continuous time-variant nonlinear function and $\sigma(x) : \Omega \to R^L$ is a neural network base function, where $L = l_1 + l_2 + \cdots + l_n$. Define

$$\varepsilon(x,t) = \begin{cases} \varepsilon_1(x,t) & 0 \le t \le t_1 + \delta_1 \\ \varepsilon_2(x,t) & t_1 + \delta_1 \le t \le t_2 + \delta_2 \\ & \vdots \\ \varepsilon_n(x,t) & t_{n-1} + \delta_{n-1} \le t \le T \end{cases}$$

Clearly, for $\forall x \in \Omega$, $|t - t_i| < \delta_i$ and we have $|\varepsilon_i(x,t)| < \varepsilon_{lf}$. Based on the definitions of $W^*(t)$, $\sigma(x)$ and $V_i^*(t)$, we can obtain

$$F(x,t) = -\alpha x(t) + W^*(t)\sigma(x(t) + \varepsilon(x,t)) \quad x \in \Omega \quad t \in [0,T]$$

and $|\varepsilon_i(x,t)| < \varepsilon_{lf}$.

From the theoretical proof, we can see that the essence of the approximation is that any finite time trajectory of a given n-dimensional dynamical system can be approximately modeled by the internal state of the output units of a continuous time recurrent neural network.

3. **Algorithm Design.** In the process of multi-layer feedback neural network training weight matrix, each variable of nonlinear continuous system is related to two independent dynamic factors: continuous time and the number of training iterations. Using the expression form of 2-D continuous-discrete system, Equation (1) can be expressed as

$$\frac{\partial z(t,k)}{\partial t} = -\alpha z(t,k) + W(t,k)\sigma(z(t,k)) \tag{6}$$

The training iteration of connection weight matrix of nonlinear continuous system can be defined as

$$W(t,k+1) = W(t,k) + \Delta W(t,k) \tag{7}$$

Assume that the difference between the real value and the simulated value is the amount of deviation e

$$e(t,k) = y(t) - x(t,k) \tag{8}$$

where $y(t) \in R^L$.

Let the initial condition of the weight matrix be

$$x(0,k) = x(0) = y(0) \quad k = 1, 2, \ldots \tag{9}$$

and

$$\eta(t,k) = \int_0^t [x(\tau,k+1) - x(\tau,k)]d\tau \tag{10}$$

$$\begin{aligned} & e(t,k+1) - e(t,k) \\ = \; & x(t,k) - x(t,k+1) \\ = \; & \alpha\eta(t,k) - \int_0^t [f(W_1(\tau,k+1), x(\tau,k+1)) - f(W_1(\tau,k), x(\tau,k))]d\tau \end{aligned} \tag{11}$$

So we can get

$$\Delta W_1(t,k) = \left(\frac{\partial e(t,k)}{\partial t} + W_1(t,k)(\sigma(x(t,k)) - \sigma(x(t,k+1)))\right)$$

$$\times \left((\sigma(x(t,k+1)))^T \sigma(x(t,k+1))\right)^{-1} \times (\sigma(x(t,k+1)))^T \tag{12}$$

4. **Simulation Results.** To further verify time-variant neural networks in Section 2 to a dynamical time-variant system. In this section, we present a numerical example with simulation result to demonstrate the effectiveness of the proposed results in the previous section. Experimental studies that apply the time-variant system were considered as the following mathematical model:

$$\begin{cases} \dfrac{dp_1(t)}{dt} = t \cdot p_2(t) \\ \dfrac{dp_2(t)}{dt} = -p_1^2(t) - p_2(t) \end{cases} \tag{13}$$

with initial conditions $p_1(0) = 0$, $p_2(0) = 1$.

The training of Equation (1) was used for the approximation of the time-variant dynamical system (13).

In the presented results, the parameter α of Equation (1) was set to 0.01, and the basis function $\tanh(x)$ was chosen for $\sigma(x)$. For convenience, both the time-variant system (13) and Equation (1) were discretized in the simulation with a sampling interval of 0.01. The real-time modeling algorithm is iterated twice at each time point.

By comparative Figure 1 and Figure 2, it can be seen that the nonlinear time-variant system (13) can be approximated by Equation (1) with a very high degree of accuracy.

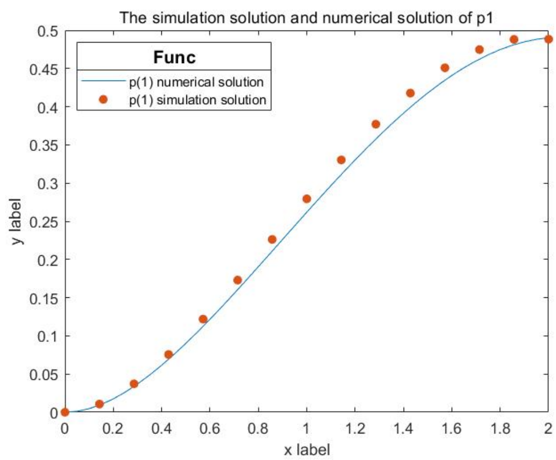

FIGURE 1. The simulation solution and numerical solution of p_1

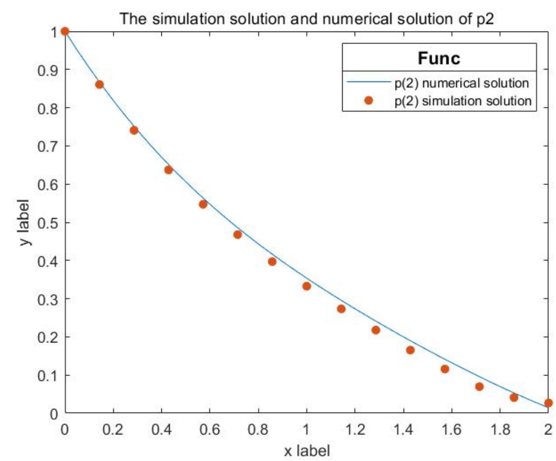

FIGURE 2. The simulation solution and numerical solution of p_2

This modeling algorithm has a good real-time performance. At each time point, it usually requires only a few iterative training operations to achieve the modeling accuracy required by the nonlinear continuous system we want to simulate, and this method has a small amount of calculation. Moreover, the results show that time-variant dynamical systems can be approximately modeled by time-variant neural networks.

5. **Conclusions.** The main contribution of this paper is to prove that time-variant neural networks can be used to uniformly approximate time-variant dynamical systems. The proof used in this paper is constructive. The simulation presented here illustrates the approximation capability of a time-variant neural network for nonlinear time-variant system. Our method solved the problem of nonlinear system identification based on time-varying neural networks. However, it should be mentioned that diffusion or time-variant delay of nonlinear systems is usually unavoidable, and this can be described by diffusion neural network or time-variant delay neural network, which makes one of possible directions in the future.

Acknowledgment. This work is supported by Lianyungang Postdoctoral Foundation.

REFERENCES

[1] M. Forgione and D. Piga, Model structures and fitting criteria for system identification with neural networks, *2020 IEEE 14th International Conference on Application of Information and Communication Technologies (AICT)*, pp.1-17, 2020.
[2] M. Forgione and D. Piga, Continuous-time system identification with neural networks: Model structures and fitting criteria, *European Journal of Control*, vol.59, pp.69-81, 2021.
[3] I. A. Aljamaan, M. M. Al-Dhaifallah and D. T. Westwick, Hammerstein Box-Jenkins system identification of the cascaded tanks benchmark system, *Mathematical Problems in Engineering*, vol.2021, no.9, pp.1-8, 2021.
[4] R. K. Raji, M. Adjeisah, X. Miao and A. Wang, A novel respiration pattern biometric prediction system based on artificial neural network, *Sensor Review*, vol.40, no.1, pp.8-16, 2020.
[5] G. da S. Lima, D. R. Porto, A. J. de Oliveira et al., Intelligent control of a single-link flexible manipulator using sliding modes and artificial neural networks, *Electronics Letters*, vol.57, no.23, pp.869-872, 2022.
[6] X. Guan, S. Lou, H. Li et al., Intelligent control of quad-rotor aircrafts with a STM32 microcontroller using deep neural networks, *Industrial Robot*, vol.48, no.5, pp.700-709, 2021.
[7] A. Taniguchi, S. Mu, S. Shibata et al., An intelligent PID control using neural networks for pneumatic servo systems, *2020 International Symposium on Computer, Consumer and Control (IS3C)*, 2020.
[8] H. Iiduka, Appropriate learning rates of adaptive learning rate optimization algorithms for training deep neural networks, *IEEE Transactions on Cybernetics*, vol.52, no.12, pp.13250-13261, 2020.
[9] T. J. Rani, T. J. Vumesh, P. Saiteja et al., SMS spam detection framework using machine learning algorithms and neural networks, *International Journal of Computer Science and Mobile Computing*, vol.10, no.6, pp.10-19, 2021.
[10] X. Meng, H. Babaee and G. E. Karniadakis, Multi-fidelity Bayesian neural networks: Algorithms and applications, *Journal of Computational Physics*, vol.438, pp.1-16, 2021.
[11] Y. Liu, S. Liu, Y. Wang et al., A survey of stochastic computing neural networks for machine learning applications, *IEEE Transactions on Neural Networks and Learning Systems*, vol.32, no.7, pp.2809-2824, 2020.
[12] E. Checcucci, S. D. Cillis, S. Granato et al., Applications of neural networks in urology: A systematic review, *Current Opinion in Urology*, vol.30, no.6, pp.788-807, 2020.
[13] Y. Nakamura and M. Nakagawa, Approximation capability of continuous time recurrent neural networks for non-autonomous dynamical systems, *Artificial Neural Networks – ICANN 2009*, pp.593-602, 2009.
[14] K. I. Funahashi and Y. Nakamura, Approximation of dynamical systems by continuous time recurrent neural networks, *Neural Networks*, vol.6, no.6, pp.801-806, 1993.
[15] G. Vasu, M. S. Kumar and M. R. Raju, Internal model control design based on approximation of linear discrete dynamical systems, *Applied Mathematical Modelling*, vol.97, pp.683-700, 2021.
[16] K. Funahashi, On the approximate realization of continuous mappings by neural networks, *Neural Networks*, vol.2, no.3, pp.183-192. 1989.