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THE ISOMORPHISM THEOREMS FOR HILBERT ALGEBRAS

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Abstract. *In this paper, we provide the necessary basic properties of homomorphisms in Hilbert algebras. The main theorem of this paper, namely the Fundamental Theorem of Homomorphisms, is constructed using the quotient Hilbert algebra of the congruence induced by an ideal. We also give an application of the theorem to the first, second, and third isomorphism theorems in Hilbert algebras.*

Keywords: Hilbert algebra, Quotient Hilbert algebra, Ideal, Fundamental Theorem of Homomorphisms, First, second, and third isomorphism theorems

1. **Introduction.** Among many algebraic structures, algebras of logic form an important class of algebras. The concept of Hilbert algebras was introduced by Diego [1]. Diego proved that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by Busneag [2, 3] and Jun [4], and some of their filters forming deductive systems were recognized. Dudek [5] considered the fuzzification of subalgebras/ideals and deductive systems in Hilbert algebras.

Isomorphism theorems in various algebraic systems are important and are being studied by mathematicians continuously and extensively, which we will discuss below. In 1998, Jun et al. [6] proved isomorphism theorems using the Chinese Remainder Theorem in BCI-algebras. In 2001, Park et al. [7] proved isomorphism theorems of IS-algebras. In 2004, Hao and Li [8] proved isomorphism theorems by using the concept of ideals in BCI-algebras. In 2006, Kim [9] proved the first isomorphism theorem for KS-semigroups. In 2008, Kim and Kim [10] proved isomorphism theorems of BG-algebras. In 2009, Paradero-Vilela and Cawi [11] characterized ideals of KS-semigroups and proved the first isomorphism theorem for KS-semigroups. In 2011, Keawrahun and Leerawat [12] proved isomorphism theorems for SU-semigroups. In 2012, Asawasamrit [13] proved isomorphism theorems of KK-algebras. In 2019, Iampan [14] proved isomorphism theorems of UP-algebras. Bejarasco and Gonzaga [15] introduced the notion of AB-homomorphism of AB-algebras and investigated the first and third isomorphism theorems. In 2020, Abed [16] introduced homomorphisms of BZ-algebras and investigated their properties. They also proved isomorphism theorems for BZ-algebras. In 2021, Chaudhry et al. [17] proved isomorphism theorems for generalized d-algebras. Emmanuel [18] established the Fundamental Theorem of Homomorphisms of torian algebras and proved isomorphism theorems of torian algebras. In 2022, Sriponpaew and Sassanapitax [19] introduced the notion of

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weak AB-algebras, which is a generalization of BCC-algebras. They proved the fundamental theorems of isomorphism for weak AB-algebras. In 2023, Bolima and Fuentes [20] proved the first and third isomorphism theorems for dual B-algebras. The reviewed articles inspired us to study isomorphism theorems in Hilbert algebras.

In this paper, we describe the essential fundamental properties of homomorphisms in Hilbert algebras. The Fundamental Theorem of Homomorphisms in Hilbert algebras, the central theorem of this paper, is derived from the quotient Hilbert algebra of the congruence induced by an ideal, and a diagram of the theorem is presented. The theorem is also applied to the first, second, and third isomorphism theorems for Hilbert algebras.

2. **Preliminaries.** The concept of Hilbert algebras, as it was initially introduced by Diego [1] in 1966, will be reviewed initially.

Definition 2.1. [1] *A Hilbert algebra is a triplet with the formula $X = (X, \cdot, 1)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of X that is true according to the axioms stated below:*

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

In [5], the following conclusion was established.

Lemma 2.1. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x \cdot x = 1)$,
(2) $(\forall x \in X)(1 \cdot x = x)$,
(3) $(\forall x \in X)(x \cdot 1 = 1)$,
(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$,
(5) $(\forall x, y, z \in X)((x \cdot z) \cdot ((z \cdot y) \cdot (x \cdot y)) = 1)$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.2. [21] *A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a subalgebra of X if $x \cdot y \in D$ for all $x, y \in D$.*

Definition 2.3. [22] *A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called an ideal of X if the following conditions hold:*

(1) $1 \in D$,
(2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$,
(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

Definition 2.4. [22] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra and B an ideal of X. Define the binary relation $\sim_B$ on X as follows:*

$$(\forall x, y \in X)\big(x \sim_B y \Leftrightarrow x \cdot y \in B \text{ and } y \cdot x \in B\big). \tag{1}$$

Definition 2.5. [22] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. An equivalence relation ρ on X is called a congruence if*

$$(\forall x, y, z \in X)(x\rho y \Rightarrow x \cdot z \rho y \cdot z \text{ and } z \cdot x \rho z \cdot y). \tag{2}$$

Lemma 2.2. [22] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. An equivalence relation ρ on X is a congruence if and only if*

$$(\forall x, y, u, v \in X)(x\rho y \text{ and } u\rho v \Rightarrow x \cdot u \rho y \cdot v). \tag{3}$$

Let $X = (X, \cdot, 1)$ be a Hilbert algebra and ρ a congruence on X. If $x \in X$, then the ρ-class of x is the $(x)_{\sim_\rho}$ defined as follows: $(x)_{\sim_\rho} = \{y \in X : y\rho x\}$. Then the set of all ρ-classes is called the quotient set of X by ρ and is denoted by X/ρ. That is, $X/\rho = \{(x)_{\sim_\rho} : x \in X\}$.

Theorem 2.1. [22] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra and B an ideal of X. Then $(X/\sim_B, *, (0)_{\sim_B})$ is a Hilbert algebra under the $*$ multiplication defined by $(x)_{\sim_B} * (y)_{\sim_B} = (x * y)_{\sim_B}$ for all $x, y \in X$, called the quotient Hilbert algebra of X induced by the congruence $\sim_B$.*

Theorem 2.2. [22] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra and B an ideal of X. Then the mapping $\pi_B : X \to X/\sim_B$ defined by $\pi_B(x) = (x)_{\sim_B}$ for all $x \in X$ is an epimorphism, called the natural projection from X to $X/\sim_B$.*

Proposition 2.1. [22] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra and B an ideal of X. Then $\sim_B$ is a congruence on X.*

Theorem 2.3. [22] *Let $X = (X, \cdot, 1)$ be a Hilbert algebra and B an ideal of X. Then the following statements hold:*

(1) the $\sim_B$-class $(0)_{\sim_B}$ is an ideal and a subalgebra of X in which $B = (0)_{\sim_B}$,

(2) a $\sim_B$-class $(x)_{\sim_B}$ is an ideal of X if and only if $x \in B$,

(3) a $\sim_B$-class $(x)_{\sim_B}$ is a subalgebra of X if and only if $x \in B$,

(4) $(X/\sim_B, \star, (0)_{\sim_B})$ is a Hilbert algebra under the $\star$ multiplication defined by $(x)_{\sim_B} \star (y)_{\sim_B} = (x \star y)_{\sim_B}$ for all $x, y \in X$, called the quotient Hilbert algebra of X induced by the congruence $\sim_B$.

3. **Main Results.** In this section, we construct the Fundamental Theorem of Homomorphisms in Hilbert algebras. We also give an application of the theorem to the first, second, and third isomorphism theorems in Hilbert algebras.

Definition 3.1. *Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras. A function $f : A \to B$ is called a homomorphism if $f(x \cdot y) = f(x) \star f(y)$ for all $x, y \in A$. A homomorphism $f : A \to B$ is called*

(1) an epimorphism if f is surjective,

(2) a monomorphism if f is injective,

(3) an isomorphism if f is bijective. Moreover, we say A is isomorphic to B, symbolically, $A \cong B$, if there is an isomorphism from A to B.

Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras. Let $f : A \to B$ be a function and let U be a nonempty subset of A and V of B. The set $\{f(x) : x \in U\}$ is called the image of U under f, denoted by $f(U)$. In particular, $f(A)$ is called the image of f, denoted by $\mathrm{Im}(f)$. Dually, the set $\{x \in A : f(x) \in V\}$ is said the inverse image of V under f, symbolically, $f^{-1}(V)$. Especially, we say $f^{-1}(\{1_B\})$ is the kernel of f, written by $\mathrm{Ker}(f)$. That is, $\mathrm{Im}(f) = \{f(x) : x \in A\}$ and $\mathrm{Ker}(f) = \{x \in A : f(x) = 1_B\}$.

In fact, it is easy to show the following theorem.

Theorem 3.1. *Let A, B, and C be Hilbert algebras. Then the following statements hold:*

(1) the identity mapping is an isomorphism,

(2) if $f : A \to B$ is an isomorphism, then $f^{-1} : B \to A$ is an isomorphism,

(3) if $f : A \to B$ and $g : B \to C$ are isomorphisms, then $g \circ f : A \to C$ is an isomorphism.

Theorem 3.2. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra and B an ideal of X. Then the mapping $\pi_B : A \to A/\sim_B$ defined by $\pi_B(x) = (x)_{\sim_B}$ for all $x \in A$ is an epimorphism, called the natural projection from A to $A/\sim_B$.*

Proof: Let $x, y \in X$ be such that $x = y$. Then $(x)_{\sim_B} = (y)_{\sim_B}$, so $\pi_B(x) = \pi_B(y)$. Thus, π_B is well-defined. Note that by the definition of π_B, we have π_B is surjective. Let $x, y \in X$. Then $\pi_B(x \cdot y) = (x \cdot y)_{\sim_B} = (x)_{\sim_B} \star (y)_{\sim_B} = \pi_B(x) \star \pi_B(y)$. Thus, π_B is a homomorphism. Hence, π_B is an epimorphism. □

Theorem 3.3. *Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and let $f : A \to B$ be a homomorphism. Then the following statements hold:*

(1) $f(1_A) = 1_B$,
(2) for any $x, y \in A$, if $x \leq y$, then $f(x) \leq f(y)$,
(3) if C is a subalgebra of A, then the image $f(C)$ is a subalgebra of B. In particular, $\mathrm{Im}(f)$ *is a subalgebra of B,*
(4) if D is a subalgebra of B, then the inverse image $f^{-1}(D)$ is a subalgebra of A. In particular, $\mathrm{Ker}(f)$ *is a subalgebra of A,*
(5) if C is an ideal of A, then the image $f(C)$ is an ideal of $f(A)$,
(6) if D is an ideal of B, then the inverse image $f^{-1}(D)$ is an ideal of A. In particular, $\mathrm{Ker}(f)$ *is an ideal of A,*
(7) $\mathrm{Ker}(f) = \{1_A\}$ *if and only if f is injective.*

Proof:

(1) By Lemma 2.1 (1), we have $f(1_A) = f(1_A \cdot 1_A) = f(1_A) \star f(1_A) = 1_B$.

(2) If $x \leq y$, then $x \cdot y = 1_A$. By (1), we have $f(x) \star f(y) = f(x \cdot y) = f(1_A) = 1_B$. Hence, $f(x) \leq f(y)$.

(3) Assume that C is a subalgebra of A. Since $1_A \in C$, we have $f(1_A) \in f(C) \neq \emptyset$. Let $a, b \in f(C)$. Then $f(x) = a$ and $f(y) = b$ for some $x, y \in C$. Since C is closed under the $\cdot$ multiplication on A, we get $a \star b = f(x) \star f(y) = f(x \cdot y) \in f(C)$. Hence, $f(C)$ is a subalgebra of B. In particular, since A is a subalgebra of A, we obtain $\mathrm{Im}(f) = f(A)$ is a subalgebra of B.

(4) Assume that D is a subalgebra of B. Since $1_B \in D$, it follows from (1) that $1_A \in f^{-1}(D) \neq \emptyset$. Let $x, y \in f^{-1}(D)$. Then $f(x), f(y) \in D$. Since D is closed under the $\star$ multiplication on B, we get $f(x \cdot y) = f(x) \star f(y) \in D$. Thus, $x \cdot y \in f^{-1}(D)$, it follows that $f^{-1}(D)$ is a subalgebra of A. In particular, since $\{1_B\}$ is a subalgebra of B, we obtain $\mathrm{Ker}(f) = f^{-1}(\{1_B\})$ is a subalgebra of A.

(5) Assume that C is an ideal of A. Since $1_A \in C$ and (1), we have $1_B = f(1_A) \in f(C)$. Let $a, b \in f(A)$ be such that $b \in f(C)$. Then $f(y) = b$ for some $y \in C$, and $f(x) = a$ for some $x \in A$. Then $a \star b = f(x) \star f(y) = f(x \cdot y) \in f(C)$. Let $a, b, c \in f(A)$ be such that $a, b \in f(C)$. Then $f(y_1) = a$ and $f(y_2) = b$ for some $y_1, y_2 \in C$, and $f(x) = c$ for some $x \in A$. Then $(a \star (b \star c)) \star c = (f(y_1) \star (f(y_2) \star f(x))) \star f(x) = f((y_1 \cdot (y_2 \cdot x)) \cdot x) \in f(C)$, proving $f(C)$ is an ideal of $f(A)$.

(6) Assume that D is an ideal of B. Since $1_B \in D$ and (1), we have $f(1_A) = 1_B \in D$. Thus, $1_A \in f^{-1}(D)$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in f^{-1}(D)$ and $y \in f^{-1}(D)$. Then $f(x \cdot (y \cdot z)) \in D$ and $f(y) \in D$. Since f is a homomorphism, we have $f(x) \star (f(y) \star f(z)) = f(x \cdot (y \cdot z)) \in D$. Since D is an ideal of B and $f(y) \in D$, we have $f(x \cdot z) = f(x) \star f(z) \in D$. Thus, $x \cdot z \in f^{-1}(D)$. Hence, $f^{-1}(D)$ is an ideal of A. In particular, since $\{1_B\}$ is an ideal of B, we obtain $\mathrm{Ker}(f) = f^{-1}(\{1_B\})$ is an ideal of A.

(7) Assume that $\mathrm{Ker}(f) = \{1_A\}$. Let $x, y \in A$ be such that $f(x) = f(y)$. By Lemma 2.1 (1), we have $f(x \cdot y) = f(x) \star f(y) = f(y) \star f(y) = 1_B$ and $f(y \cdot x) = f(y) \star f(x) = f(y) \star f(y) = 1_B$. Thus, $x \cdot y, y \cdot x \in \mathrm{Ker}(f) = \{1_A\}$, so $x \cdot y = y \cdot x = 1_A$. Thus, $x = y$. Hence, f is injective.

Conversely, assume that f is injective. By (1), we obtain $\{1_A\} \subseteq \mathrm{Ker}(f)$. Let $x \in \mathrm{Ker}(f)$. Then $f(x) = 1_B = f(1_A)$, so $x = 1_A$ because f is injective. Hence, $\mathrm{Ker}(f) = \{1_A\}$.
□

Theorem 3.4. *(Fundamental Theorem of Homomorphisms) Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and $f : A \to B$ a homomorphism. Then there exists uniquely a homomorphism φ from $A/\sim_{\mathrm{Ker}(f)}$ to B such that $f = \varphi \circ \pi_{\mathrm{Ker}(f)}$. Moreover,*

(1) $\pi_{\mathrm{Ker}(f)}$ is an epimorphism and φ a monomorphism,

(2) f is an epimorphism if and only if φ is an isomorphism. We get the following diagram.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\pi_{\mathrm{Ker}(f)}"'] & & B \\
& A/\sim_{\mathrm{Ker}(f)} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Proof: Put $K = \mathrm{Ker}(f)$. By Theorem 3.3 (6), we have K is an ideal of A. It follows from Theorem 2.3 (4) that $(A/\sim_K, \star, (1_A)_{\sim_K})$ is a Hilbert algebra. Define $\varphi : A/\sim_K \to B$, $(x)_{\sim_K} \to f(x)$. Let $(x)_{\sim_K}, (y)_{\sim_K} \in A/\sim_K$ be such that $(x)_{\sim_K} = (y)_{\sim_K}$. Then $x \sim_K y$, so $x \cdot y \in K$ and $y \cdot x \in K$. Thus, $f(x) \star f(y) = f(x \cdot y) = 1_B$ and $f(y) \star f(x) = f(y \cdot x) = 1_B$. Thus, $f(x) = f(y)$ and so $\varphi\big((x)_{\sim_K}\big) = \varphi\big((y)_{\sim_K}\big)$. Thus, φ is a mapping. For any $x, y \in A$, we see that $\varphi\big((x)_{\sim_K} \star (y)_{\sim_K}\big) = \varphi\big((x \cdot y)_{\sim_K}\big) = f(x \cdot y) = f(x) \star f(y) = \varphi\big((x)_{\sim_K}\big) \star \varphi\big((y)_{\sim_K}\big)$. Thus, φ is a homomorphism. Also, since $(\varphi \circ \pi_K)(x) = \varphi(\pi_K(x)) = \varphi\big((x)_{\sim_K}\big) = f(x)$ for all $x \in A$, we obtain $f = \varphi \circ \pi_K$. We have shown the existence. Let φ' be a mapping from $A/\sim_K$ to B such that $f = \varphi' \circ \pi_K$. Then for any $(x)_{\sim_K} \in A/\sim_K$, we have $\varphi'\big((x)_{\sim_K}\big) = \varphi'(\pi_K(x)) = (\varphi' \circ \pi_K)(x) = f(x) = (\varphi \circ \pi_K)(x) = \varphi(\pi_K(x)) = \varphi\big((x)_{\sim_K}\big)$. Hence, $\varphi = \varphi'$, showing the uniqueness.

(1) By Theorem 3.2, we have π_K is an epimorphism. Also, let $(x)_{\sim_K}, (y)_{\sim_K} \in A/\sim_K$ be such that $\varphi\big((x)_{\sim_K}\big) = \varphi\big((y)_{\sim_K}\big)$. Then $f(x) = f(y)$ and it follows from Lemma 2.1 (1) that $f(x \cdot y) = f(x) \star f(y) = f(y) \star f(y) = 1_B$, that is, $x \cdot y \in K$. Similarly, $y \cdot x \in K$. Hence, $x \sim_K y$ and $(x)_{\sim_K} = (y)_{\sim_K}$. Therefore, φ is a monomorphism.

(2) Assume that f is an epimorphism. By (1), it suffices to prove φ is surjective. Let $y \in B$. Then there exists $x \in A$ such that $f(x) = y$. Thus, $y = f(x) = \varphi\big((x)_{\sim_K}\big)$, so φ is surjective. Hence, φ is an isomorphism.

Conversely, assume that φ is an isomorphism. Then φ is surjective. Let $y \in B$. Then there exists $(x)_{\sim_K} \in A/\sim_K$ such that $\varphi\big((x)_{\sim_K}\big) = y$. Thus, $f(x) = \varphi\big((x)_{\sim_K}\big) = y$, so f is surjective. Hence, f is an epimorphism. □

Theorem 3.5. *(First Isomorphism Theorem) Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras and $f : A \to B$ a homomorphism. Then $A/\sim_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f)$.*

Proof: By Theorem 3.3 (3), we have $\mathrm{Im}(f)$ is a subalgebra of B. Thus, $f : A \to \mathrm{Im}(f)$ is an epimorphism. Applying Theorem 3.4 (2), we obtain $A/\sim_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f)$. □

Lemma 3.1. *Let $A = (A, \cdot, 1_A)$ be a Hilbert algebra, H a subalgebra of A, and K an ideal of A. Denote $HK = \bigcup_{h \in H} (h)_{\sim_K}$. Then HK is a subalgebra of A.*

Proof: Clearly, $\emptyset \neq HK \subseteq A$. Let $a, b \in HK$. Then $a \in (x)_{\sim_K}$ and $b \in (y)_{\sim_K}$ for some $x, y \in H$, so $(a)_{\sim_K} = (x)_{\sim_K}$ and $(b)_{\sim_K} = (y)_{\sim_K}$. Thus, $(a \cdot b)_{\sim_K} = (a)_{\sim_K} \star (b)_{\sim_K} = (x)_{\sim_K} \star (y)_{\sim_K} = (x \cdot y)_{\sim_K}$, so $a \cdot b \in (x \cdot y)_{\sim_K}$. Since $x, y \in H$, it follows that $x \cdot y \in H$. Thus, $a \cdot b \in (x \cdot y)_{\sim_K} \subseteq HK$. Hence, HK is a subalgebra of A. □

Theorem 3.6. *(Second Isomorphism Theorem) Let $A = (A, \cdot, 1_A)$ be a Hilbert algebra, H a subalgebra of A, and K an ideal of A. Denote $HK/\sim_K = \big\{(x)_{\sim_K} : x \in HK\big\}$. Then $H/\sim_{H \cap K} \cong HK/\sim_K$.*

Proof: By Lemma 3.1, HK is a subalgebra of A. Then it is easy to check that $HK/\sim_K$ is a subalgebra of $A/\sim_K$. Thus, $\big(HK/\sim_K, \star, (1_A)_{\sim_K}\big)$ itself is a Hilbert algebra. Also, it is obvious that $H \subseteq HK$, then $f : H \to HK/\sim_K$, $x \mapsto (x)_{\sim_K}$, is a mapping. For any $x, y \in H$, we have $f(x \cdot y) = (x \cdot y)_{\sim_K} = (x)_{\sim_K} \star (y)_{\sim_K} = f(x) \star f(y)$. Thus, f is a homomorphism. We shall show that f is an epimorphism with $\mathrm{Ker}(f) = H \cap K$. For any $(x)_{\sim_K} \in HK/\sim_K$, we have $x \in HK = \bigcup_{h \in H}(h)_{\sim_K}$. Then there exists $h \in H$ such that $x \in (h)_{\sim_K}$ and so $(x)_{\sim_K} = (h)_{\sim_K}$. Thus, $f(h) = (h)_{\sim_K} = (x)_{\sim_K}$. Therefore, f is an epimorphism. Also, for any $h \in H$, if $h \in \mathrm{Ker}(f)$, then $f(h) = (1_A)_{\sim_K}$. Since $f(h) = (h)_{\sim_K}$, we obtain $(h)_{\sim_K} = (1_A)_{\sim_K}$. By (1), we have $h = 1_A \cdot h \in K$. Thus, $h \in H \cap K$, that is, $\mathrm{Ker}(f) \subseteq H \cap K$. On the other hand, if $h \in H \cap K$, by $h \in H$, $f(h)$ is well-defined, by $h \in K$ and $1_A \in K$, $h \cdot 1_A \in K$ and $1_A \cdot h \in K$. By (1), we have $h \sim_K 1_A$ and so $(h)_{\sim_K} = (1_A)_{\sim_K}$. Thus, $f(h) = (h)_{\sim_K} = (1_A)_{\sim_K}$. So, $h \in \mathrm{Ker}(f)$, that is, $H \cap K \subseteq \mathrm{Ker}(f)$. Therefore, $\mathrm{Ker}(f) = H \cap K$. By Theorem 3.5, we have $H/\sim_{H\cap K} \cong HK/\sim_K$. □

Theorem 3.7. *(Third Isomorphism Theorem) Let $A = (A, \cdot, 1_A)$ be a Hilbert algebra and H and K ideals of A with $H \subseteq K$. Then $(A/\sim_H)/\sim (K/\sim_H) \cong A/\sim_K$.*

Proof: By Theorem 2.3 (4), we obtain $\big(A/\sim_K, \star, (1_A)_{\sim_K}\big)$ and $\big(A/\sim_H, \star', (1_A)_{\sim_H}\big)$ are Hilbert algebras. Define $f : A/\sim_H \to A/\sim_K$, $(x)_{\sim_H} \mapsto (x)_{\sim_K}$. For any $x, y \in A$, if $(x)_{\sim_H} = (y)_{\sim_H}$, then $x \cdot y, y \cdot x \in H$. Since $H \subseteq K$, we obtain $x \cdot y, y \cdot x \in K$. Thus, $(x)_{\sim_K} = (y)_{\sim_K}$, so $f\big((x)_{\sim_H}\big) = f\big((y)_{\sim_H}\big)$. Thus, f is a mapping. Also, for any $x, y \in A$, we see that $f\big((x)_{\sim_H} \star (y)_{\sim_H}\big) = f\big((x \cdot y)_{\sim_H}\big) = (x \cdot y)_{\sim_K} = (x)_{\sim_K} \star (y)_{\sim_K} = f\big((x)_{\sim_H}\big) \star f\big((y)_{\sim_H}\big)$. Thus, f is a homomorphism. Clearly, f is surjective. Hence, f is an epimorphism. We shall show that $\mathrm{Ker}(f) = K/\sim_H$. In fact,

$$\begin{aligned}
\mathrm{Ker}(f) &= \big\{(x)_{\sim_H} \in A/\sim_H : f\big((x)_{\sim_H}\big) = (1_A)_{\sim_K}\big\} \\
&= \big\{(x)_{\sim_H} \in A/\sim_H : (x)_{\sim_K} = (1_A)_{\sim_K}\big\} \\
&= \big\{(x)_{\sim_H} \in A/\sim_H : x = 1_A \cdot x \in K\big\} \\
&= K/\sim_H.
\end{aligned}$$

By Theorem 3.5, we have $(A/\sim_H)/\sim (K/\sim_H) \cong A/\sim_K$. □

4. **Conclusion.** For this paper, we have given several important fundamental properties of homomorphisms in Hilbert algebras. We have constructed the Fundamental Theorem of Homomorphisms in Hilbert algebras using the quotient Hilbert algebra of the congruence induced by an ideal. Finally, we derived the first, second, and third isomorphism theorems in Hilbert algebra from the Fundamental Theorem of Homomorphisms.

To expand on the results of this paper, future research will focus on finding isomorphism theorems of obic algebras and torian algebras, which were introduced by Emmanuel [23, 24].

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