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INTUITIONISTIC HESITANT FUZZY DEDUCTIVE SYSTEMS OF HILBERT ALGEBRAS

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Abstract. *In this paper, the concept of intuitionistic hesitant fuzzy deductive systems of Hilbert algebras is introduced. The relationship between intuitionistic hesitant fuzzy deductive systems and their π-level subsets is described. Moreover, the homomorphic pre-images of intuitionistic hesitant fuzzy deductive systems of Hilbert algebras are also studied and some related properties are investigated.*

Keywords: Hilbert algebra, Intuitionistic hesitant fuzzy deductive system, π-level subset, Homomorphic pre-image

1. **Introduction.** Zadeh introduced the idea of fuzzy sets in [1]. Numerous academics have studied fuzzy set theory because it has numerous practical applications. Numerous studies on the extensions of fuzzy sets were undertaken after the notion of fuzzy sets was first introduced; one of these studies is the intuitionistic fuzzy set developed by Atanassov [2]. In [3, 4, 5, 6], it has been addressed how fuzzy sets may be integrated with various uncertainty techniques, such as soft sets and rough sets. The concept of hesitant fuzzy sets, which is a function from a reference set to a power set of the unit interval, was presented in 2009-2010 by Torra and Narukawa [7, 8]. The second way that the concept of fuzzy sets is extended is via the concept of hesitant fuzzy sets. The hesitant fuzzy set theories created by Torra and others have a wide range of uses, both in and outside of the field of mathematics. After Torra and Narukawa [7, 8] proposed the idea of hesitant fuzzy sets, several studies were done to generalize the idea and apply it to different logical algebras. For example, in 2012, Zhu et al. [9] developed the idea of dual hesitant fuzzy sets, which is a novel extension of fuzzy sets. The concepts of hesitant fuzzy subalgebras

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and hesitant fuzzy ideals of BCK/BCI-algebras were presented by Jun and Ahn [10] in 2016. Mosrijai et al. [11] introduced the notion of hesitant fuzzy sets on UP-algebras in 2017. In understanding the various logical algebras, the concepts of hesitant fuzzy subalgebras, hesitant fuzzy filters, and hesitant fuzzy ideals are crucial. The concept of Hilbert algebras was introduced in early 50-ties by Henkin [12] for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by Diego [13] from algebraic point of view. Diego proved that Hilbert algebras form a variety which is locally finite. Busneag [14, 15] and Jun [16] both addressed Hilbert algebras, and it was realized that certainty of their filters formed deductive systems. In Hilbert algebras, Dudek [17, 18, 19] explored the fuzzification of subalgebras, ideals, and deductive systems. The concept of deductive systems in Hilbert algebras is studied in many dimensions, for example, Iampan et al. [20, 21] studied anti-hesitant fuzzy deductive systems and interval-valued neutrosophic deductive systems of Hilbert algebras in 2022. In 2023, Iampan et al. [22] studied anti-fuzzy deductive systems of Hilbert algebras. Based on the concept of deductive systems, we are interested in studying them from the point of view of intuitionistic hesitant fuzzy sets.

The idea of intuitionistic hesitant fuzzy deductive systems of Hilbert algebras is presented in this study. From the study, there is an interesting relationship between deductive systems and their characteristic intuitionistic hesitant fuzzy sets. There is a description of the connection between intuitionistic hesitant fuzzy deductive systems and their π-level subsets. In addition, several related features are researched together with the homomorphic pre-images of intuitionistic hesitant fuzzy deductive systems of Hilbert algebras.

2. **Preliminaries.** We will first go through the idea of Hilbert algebras as it was initially introduced by Diego [13] in 1966.

Definition 2.1. [13] *A Hilbert algebra is a triplet with the formula* $X = (X, \cdot, 1)$*, where* X *is a nonempty set,* $\cdot$ *is a binary operation, and* 1 *is a fixed member of* X *that is true according to the axioms stated below: (1)* $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$*, (2)* $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$*, (3)* $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$*.*

In [17], the following conclusion was established.

Lemma 2.1. *Let* $X = (X, \cdot, 1)$ *be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x \cdot x = 1)$*,*
(2) $(\forall x \in X)(1 \cdot x = x)$*,*
(3) $(\forall x \in X)(x \cdot 1 = 1)$*,*
(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$*.*

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by $(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1)$, which is a partial order on X with 1 as the largest element.

Definition 2.2. [19] *A nonempty subset* D *of a Hilbert algebra* $X = (X, \cdot, 1)$ *is called a deductive system of* X *if the following conditions hold: (1)* $1 \in D$ *and (2)* $(\forall x, y \in X)(x \cdot y, x \in D \Rightarrow y \in D)$*.*

Definition 2.3. [7] *A hesitant fuzzy set (HFS) on a reference set* X *is defined in terms of a function* h *that when applied to* X *return a subset of* $[0, 1]$*, that is,* $h : X \to \mathcal{P}([0, 1])$*.*

Definition 2.4. [23] *An intuitionistic hesitant fuzzy set (IHFS) on a reference set* X *is defined in the form* $\mathcal{H} = (h, k)$*, where* h *and* k *are functions that when applied to* X *return a subset of* $[0, 1]$*, that is,* $h, k : X \to \mathcal{P}([0, 1])$*.*

Definition 2.5. [7] *The complement of an HFS* h *in a reference set* X *is the HFS* $\overline{h}$ *defined by* $\overline{h}(x) = [0, 1] - h(x)$ *for all* $x \in X$*.*

Definition 2.6. [7] *The complement of an IHFS $\mathcal{H} = (h, k)$ on a reference set X is the IHFS $\overline{\mathcal{H}} = \left(\overline{k}, \overline{h}\right)$ defined by $\overline{h}(x) = [0, 1] - h(x)$ and $\overline{k}(x) = [0, 1] - k(x)$ for all $x \in X$.*

Definition 2.7. [24] *An HFS h on a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a hesitant fuzzy deductive system of X if the following conditions hold:*

$$(\forall x \in X)\left(\ h(1) \supseteq h(x)\ \right) \tag{1}$$

$$(\forall x, y \in X)\left(\ h(y) \supseteq h(x \cdot y) \cap h(x)\ \right) \tag{2}$$

3. **Main Results.** In this section, we introduce the concept of intuitionistic hesitant fuzzy deductive systems of Hilbert algebras and provide some interesting properties.

Definition 3.1. [25] *An IHFS $\mathcal{H} = (h, k)$ on a Hilbert algebra $X = (X, \cdot, 1)$ is called an intuitionistic hesitant fuzzy ideal of X if the following conditions hold:*

$$(\forall x \in X)\left(\begin{array}{c} h(1) \supseteq h(x) \\ k(1) \subseteq k(x) \end{array}\right) \tag{3}$$

$$(\forall x, y \in X)\left(\begin{array}{c} h(x \cdot y) \supseteq h(y) \\ k(x \cdot y) \subseteq k(y) \end{array}\right) \tag{4}$$

$$(\forall x, y_1, y_2 \in X)\left(\begin{array}{c} h((y_1 \cdot (y_2 \cdot x)) \cdot x) \supseteq h(y_1) \cap h(y_2) \\ k((y_1 \cdot (y_2 \cdot x)) \cdot x) \subseteq k(y_1) \cup k(y_2) \end{array}\right) \tag{5}$$

Definition 3.2. *An IHFS $\mathcal{H} = (h, k)$ on a Hilbert algebra $X = (X, \cdot, 1)$ is called an intuitionistic hesitant fuzzy deductive system of X if the following conditions hold:*

$$(\forall x \in X)\left(\begin{array}{c} h(1) \supseteq h(x) \\ k(1) \subseteq k(x) \end{array}\right) \tag{6}$$

$$(\forall x, y \in X)\left(\begin{array}{c} h(y) \supseteq h(x \cdot y) \cap h(x) \\ k(y) \subseteq k(x \cdot y) \cup k(x) \end{array}\right) \tag{7}$$

Example 3.1. *Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:*

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then X is a Hilbert algebra. We define an IHFS $\mathcal{H} = (h, k)$ on X as follows:

$$h(1) = \{0.4, 0.5, 0.7\},\ h(x) = \{0.4, 0.5\},\ h(y) = \{0.5\},\ h(z) = h(0) = \emptyset,$$
$$k(1) = \{0.2\},\ k(x) = h(y) = k(z) = k(0) = \{0.2, 0.5\}.$$

Then $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X.

Proposition 3.1. *Every intuitionistic hesitant fuzzy ideal of a Hilbert algebra $X = (X, \cdot, 1)$ is an intuitionistic hesitant fuzzy deductive system.*

Proof: Let $\mathcal{H} = (h, k)$ be an intuitionistic hesitant fuzzy ideal of X. Let $x, y \in X$. By (5), we have $h(y) = h(1 \cdot y) = h(((x \cdot y) \cdot (x \cdot y)) \cdot y) \supseteq h(x \cdot y) \cap h(x)$ and $k(y) = k(1 \cdot y) = k(((x \cdot y) \cdot (x \cdot y)) \cdot y) \subseteq k(x \cdot y) \cup k(x)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X. $\square$

Lemma 3.1. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy deductive system of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)\left(\ x \leq y \Rightarrow \left\{\begin{array}{c} h(y) \supseteq h(x) \\ k(y) \subseteq k(x) \end{array}\right.\ \right) \tag{8}$$

Proof: Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 1$ and so $h(y) \supseteq h(x \cdot y) \cap h(x) = h(1) \cap h(x) = h(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x) = k(1) \cup k(x) = k(x)$. □

Theorem 3.1. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy deductive system of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X) \left(\begin{array}{l} h(x \cdot y) = h(1) \Rightarrow h(x) \subseteq h(y) \\ k(x \cdot y) = k(1) \Rightarrow k(x) \supseteq k(y) \end{array} \right) \tag{9}$$

$$(\forall x, y, z \in X) \left(x \cdot (y \cdot z) = 1 \Rightarrow \left\{ \begin{array}{l} h(x) \cap h(y) \subseteq h(z) \\ k(x) \cup k(y) \supseteq k(z) \end{array} \right. \right) \tag{10}$$

$$(\forall x, y, z \in X) \left(\begin{array}{l} h(x \cdot z) \cap h(z \cdot y) \subseteq h(x \cdot y) \\ k(x \cdot z) \cup k(z \cdot y) \supseteq h(x \cdot y) \end{array} \right) \tag{11}$$

Proof: Let $x, y \in X$ be such that $h(x \cdot y) = h(1)$ and $k(x \cdot y) = k(1)$. Then $h(y) \supseteq h(x) \cap h(x \cdot y) = h(x) \cap h(1) = h(x)$ and $k(y) \subseteq k(x) \cup k(x \cdot y) = k(x) \cup k(1) = k(x)$. Therefore, (9) is proved.

Let $x, y, z \in X$ be such that $x \cdot (y \cdot z) = 1$. Then $x \leq y \cdot z$. By (8), we have $h(x) \subseteq h(y \cdot z)$ and $k(x) \supseteq k(y \cdot z)$. This implies that $h(x) \cap h(y) \subseteq h(y \cdot z) \cap h(y) \subseteq h(z)$ and $k(x) \cup k(y) \supseteq k(y \cdot z) \cup k(y) \supseteq k(z)$. Therefore, (10) is proved.

Let $x, y, z \in X$. By Lemma 2.1 (4), we have $(x \cdot z) \cdot ((z \cdot y) \cdot (x \cdot y)) = 1$. Then it follows from (2) that $h(x \cdot z) \cap h(z \cdot y) \subseteq h(x \cdot y)$ and $k(x \cdot z) \cup k(z \cdot y) \supseteq h(x \cdot y)$. Therefore, (11) is proved. □

Definition 3.3. *The characteristic intuitionistic hesitant fuzzy set (CIHFS) of a subset A of a set X is defined to be the structure $\chi_A = (h_{\chi_A}, k_{\chi_A})$, where*

$$h_{\chi_A}(x) = \left\{ \begin{array}{ll} [0,1] & \text{if } x \in A \\ \emptyset & \text{otherwise} \end{array} \right. \text{ and } k_{\chi_A}(x) = \left\{ \begin{array}{ll} \emptyset & \text{if } x \in A \\ [0,1] & \text{otherwise.} \end{array} \right.$$

Lemma 3.2. [25] *The constant 1 of a Hilbert algebra $X = (X, \cdot, 1)$ is in a nonempty subset B of X if and only if $h_{\chi_B}(1) \supseteq h_{\chi_B}(x)$ and $k_{\chi_B}(1) \subseteq k_{\chi_B}(x)$ for all $x \in X$.*

Theorem 3.2. *A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is a deductive system of X if and only if the CIHFS χ_D is an intuitionistic hesitant fuzzy deductive system of X.*

Proof: Assume that D is a deductive system of X. Since $1 \in D$, it follows from Lemma 3.2 that $h_{\chi_D}(1) \supseteq h_{\chi_D}(x)$ and $k_{\chi_D}(1) \subseteq k_{\chi_D}(x)$ for all $x \in X$. Next, let $x, y \in X$.

Case 1: If $x, y \in D$, then $h_{\chi_D}(x) = [0,1]$ and $h_{\chi_D}(y) = [0,1]$. Hence, $h_{\chi_D}(y) = [0,1] \supseteq h_{\chi_D}(x \cdot y) = h_{\chi_D}(x \cdot y) \cap h_{\chi_D}(x)$. Also, $k_{\chi_D}(x) = \emptyset$ and $k_{\chi_D}(y) = \emptyset$. Hence, $k_{\chi_D}(y) = \emptyset \subseteq k_{\chi_D}(x \cdot y) = k_{\chi_D}(x \cdot y) \cup k_{\chi_D}(x)$.

Case 2: If $x \notin D$ and $y \in D$, then $h_{\chi_D}(x) = \emptyset$ and $h_{\chi_D}(y) = [0,1]$. Hence, $h_{\chi_D}(y) = [0,1] \supseteq \emptyset = h_{\chi_D}(x \cdot y) \cap h_{\chi_D}(x)$. Also, $k_{\chi_D}(x) = [0,1]$ and $k_{\chi_D}(y) = \emptyset$. Hence, $k_{\chi_D}(y) = \emptyset \subseteq [0,1] = k_{\chi_D}(x \cdot y) \cup k_{\chi_D}(x)$.

Case 3: If $x \in D$ and $y \notin D$, then $h_{\chi_D}(x) = [0,1]$ and $h_{\chi_D}(y) = \emptyset$. Since D is a deductive system of X, we have $x \cdot y \notin D$ or $x \notin D$. However, $x \in D$, so $x \cdot y \notin D$. Then $h_{\chi_D}(x \cdot y) = \emptyset$. Thus, $h_{\chi_D}(y) = \emptyset \supseteq \emptyset = h_{\chi_D}(x \cdot y) \cap h_{\chi_D}(x)$. Also, $k_{\chi_D}(x) = \emptyset$, $k_{\chi_D}(y) = \emptyset$ and $k_{\chi_D}(x \cdot y) = [0,1]$. Hence, $k_{\chi_D}(y) = [0,1] \subseteq [0,1] = k_{\chi_D}(x \cdot y) \cup k_{\chi_D}(x)$.

Case 4: If $x \notin D$ and $y \notin D$, then $h_{\chi_D}(x) = \emptyset$ and $h_{\chi_D}(y) = \emptyset$. Thus, $h_{\chi_D}(y) = \emptyset \supseteq \emptyset = h_{\chi_D}(x \cdot y) \cap h_{\chi_D}(x)$. Also, $k_{\chi_D}(x) = [0,1]$ and $k_{\chi_D}(y) = [0,1]$. Hence, $k_{\chi_D}(y) = [0,1] \subseteq [0,1] = k_{\chi_D}(x \cdot y) \cup k_{\chi_D}(x)$.

Therefore, χ_D is an intuitionistic hesitant fuzzy deductive system of X.

Conversely, assume that χ_D is an intuitionistic hesitant fuzzy deductive system of X. Since $h_{\chi_D}(1) \supseteq h_{\chi_D}(x)$ and $k_{\chi_D}(1) \subseteq k_{\chi_D}(x)$ for all $x \in X$, it follows from Lemma 3.2 that $1 \in D$. Next, let $x, y \in X$ be such that $x \cdot y \in D$ and $x \in D$. Then $h_{\chi_D}(x \cdot y) = [0,1]$

and $h_{\chi_D}(x) = [0,1]$. Thus, $h_{\chi_D}(y) \supseteq h_{\chi_D}(x \cdot y) \cap h_{\chi_D}(x) = [0,1]$, so $h_{\chi_D}(y) = [0,1]$. Thus, $y \in D$ and so D is a deductive system of X. □

Definition 3.4. *Let $\mathcal{H} = (h,k)$ be an IHFS on a reference set X. The IHFSs $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are defined as $\oplus\mathcal{H} = \left(h, \overline{h}\right)$ and $\otimes\mathcal{H} = \left(\overline{k}, k\right)$.*

Theorem 3.3. *An IHFS $\mathcal{H} = (h,k)$ is an intuitionistic hesitant fuzzy deductive system of a Hilbert algebra $X = (X, \cdot, 1)$ if and only if the IHFSs $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are intuitionistic hesitant fuzzy deductive systems of X.*

Proof: Assume that $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X. Let $x \in X$. Then $\overline{h}(1) = [0,1] - h(1) \subseteq [0,1] - h(x) = \overline{h}(x)$. Let $x, y \in X$. Then $\overline{h}(y) = [0,1] - h(y) \subseteq [0,1] - (h(x \cdot y) \cap h(x)) = ([0,1] - h(x \cdot y)) \cup ([0,1] - h(x)) = \overline{h}(x \cdot y) \cup \overline{h}(x)$. Hence, $\oplus\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X.

Let $x \in X$. Then $\overline{k}(1) = [0,1] - k(1) \supseteq [0,1] - k(x) = \overline{k}(x)$. Let $x, y \in X$. Then $\overline{k}(y) = [0,1] - k(y) \supseteq [0,1] - (k(x \cdot y) \cup k(x)) = ([0,1] - k(x \cdot y)) \cap ([0,1] - k(x)) = \overline{k}(x \cdot y) \cap \overline{k}(x)$. Hence, $\otimes\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X.

Conversely, assume that $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ are intuitionistic hesitant fuzzy deductive systems of X. Then for any $x, y \in X$, we have $h(0) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$ and $k(0) \subseteq k(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X. □

Theorem 3.4. *An IHFS $\mathcal{H} = (h,k)$ is an intuitionistic hesitant fuzzy deductive system of a Hilbert algebra $X = (X, \cdot, 1)$ if and only if the HFSs h and $\overline{k}$ are hesitant fuzzy deductive systems of X.*

Proof: Assume that $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X. Then for any $x, y \in X$, we have $h(1) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$. Hence, h is a hesitant fuzzy deductive system of X. Now, for any $x, y \in X$, we have $k(1) \subseteq k(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Then $\overline{k}(1) = [0,1] - k(1) \supseteq [0,1] - k(x) = \overline{k}(x)$ and $\overline{k}(y) = [0,1] - k(y) \supseteq [0,1] - (k(x \cdot y) \cup k(x)) = [0,1] - k(x \cdot y) \cap [0,1] - k(x) = \overline{k}(x \cdot y) \cap \overline{k}(x)$. Hence, $\overline{k}$ is a hesitant fuzzy deductive system of X.

Conversely, assume that the HFSs h and $\overline{k}$ are hesitant fuzzy deductive systems of X. Then for any $x, y \in X$, we have $h(1) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$. Now, for any $x, y \in X$, we have $\overline{k}(1) \supseteq \overline{k}(x)$ and $\overline{k}(y) \supseteq \overline{k}(x \cdot y) \cap \overline{k}(x)$. Then $[0,1] - k(1) \supseteq [0,1] - k(x)$ and so $k(1) \subseteq k(x)$. Now, $[0,1] - k(y) \supseteq [0,1] - k(x \cdot y) \cap [0,1] - k(x) = [0,1] - (k(x \cdot y) \cup k(x))$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X. □

Theorem 3.5. *An IHFS $\mathcal{H} = (h,k)$ is an intuitionistic hesitant fuzzy deductive system of a Hilbert algebra $X = (X, \cdot, 1)$ if and only if the IHFS $\overline{\mathcal{H}} = \left(\overline{k}, \overline{h}\right)$ is an intuitionistic hesitant fuzzy deductive system of X.*

Proof: Assume that $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X. Then for any $x, y \in X$, we have $h(1) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$. Hence, for any $x, y \in X$, we have $\overline{h}(0) = [0,1] - h(0) \subseteq [0,1] - h(x) = \overline{h}(x)$ and $\overline{h}(y) = [0,1] - h(y) \subseteq [0,1] - (h(x \cdot y) \cap h(x)) = [0,1] - h(x \cdot y) \cup [0,1] - h(x) = \overline{h}(x \cdot y) \cup \overline{h}(x)$. Now, for any $x, y \in X$, we have $k(0) \subseteq k(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Hence, for any $x, y \in X$, we have $\overline{k}(1) = [0,1] - k(0) \supseteq [0,1] - k(x) = \overline{k}(x)$ and $\overline{k}(y) = [0,1] - k(y) \supseteq [0,1] - (k(x \cdot y) \cup k(x)) = [0,1] - k(x \cdot y) \cap [0,1] - k(x) = \overline{k}(x \cdot y) \cap \overline{k}(x)$. Hence, $\overline{\mathcal{H}} = \left(\overline{k}, \overline{h}\right)$ is an intuitionistic hesitant fuzzy deductive system of X.

Conversely, assume that the IHFS $\overline{\mathcal{H}} = \left(\overline{k}, \overline{h}\right)$ is an intuitionistic hesitant fuzzy deductive system of X. Then for any $x, y \in X$, we have $\overline{k}(1) \supseteq \overline{k}(x)$ and $\overline{k}(y) \supseteq \overline{k}(x \cdot y) \cap \overline{k}(x)$. Then $[0,1] - k(1) \supseteq [0,1] - k(x)$ and $[0,1] - k(y) \supseteq [0,1] - (k(x \cdot y) \cup k(x))$, so $k(1) \subseteq k(x)$ and $k(y) \subseteq k(x \cdot y) \cup k(x)$. Now, for any $x, y \in X$, we have $\overline{h}(1) \subseteq \overline{h}(x)$ and $\overline{h}(y) \subseteq \overline{h}(x \cdot y)$

$\cup \overline{h}(x)$. Then $[0,1] - h(1) \subseteq [0,1] - h(x)$ and $[0,1] - h(y) \supseteq [0,1] - (h(x \cdot y) \cup h(x))$, so $h(1) \supseteq h(x)$ and $h(y) \supseteq h(x \cdot y) \cap h(x)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X. □

Theorem 3.6. *If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy deductive system of a Hilbert algebra $X = (X, \cdot, 1)$, then the sets $X_h = \{x \in X \mid h(x) = h(1)\}$ and $X_k = \{x \in X \mid k(x) = k(1)\}$ are deductive systems of X.*

Proof: Clearly, $1 \in X_h \cap X_k$. Let $x, y \in X$ be such that $x, x \cdot y \in X_h$. Then $h(x) = h(1) = h(x \cdot y)$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X, we have $h(y) \supseteq h(x) \cap h(x \cdot y) = h(1)$ and so $h(y) = h(1)$. This means that $y \in X_h$. Hence, X_h is a deductive system of X. Let $x, y \in X$ be such that $x, x \cdot y \in X_h$. Then $k(x) = k(1) = k(x \cdot y)$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X, we have $k(y) \subseteq k(x) \cup k(x \cdot y) = k(1)$ and so $k(y) = k(1)$. This means that $y \in X_k$. Hence, X_k is a deductive system of X. □

Definition 3.5. *Let $h : X \to \mathcal{P}([0,1])$. For any $\pi \in \mathcal{P}([0,1])$, the set $U(h, \pi) = \{x \in X \mid h(x) \supseteq \pi\}$ is called an upper π-level subset of h. The set $L(h, \pi) = \{x \in X \mid h(x) \subseteq \pi\}$ is called a lower π-level subset of h.*

Theorem 3.7. *An IHFS $\mathcal{H} = (h, k)$ on a Hilbert algebra $X = (X, \cdot, 1)$ is an intuitionistic hesitant fuzzy deductive system of X if and only if for all $\pi \in \mathcal{P}([0,1])$, the nonempty subsets $U(h, \pi)$ and $L(k, \pi)$ of X are deductive systems.*

Proof: Assume that $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $U(h, \pi) \neq \emptyset$ and let $x \in U(h, \pi)$. Then $h(x) \supseteq \pi$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X, we have $h(1) \supseteq h(x) \supseteq \pi$. Thus, $1 \in U(h, \pi)$. Next, let $x, y \in X$ be such that $x, x \cdot y \in U(h, \pi)$. Then $h(x) \supseteq \pi$ and $h(x \cdot y) \supseteq \pi$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X, we have $h(y) \supseteq h(x \cdot y) \cap h(x) \supseteq \pi$. So $y \in U(h, \pi)$. Hence, $U(h, \pi)$ is a deductive system of X. Let $\pi \in \mathcal{P}([0,1])$ be such that $L(k, \pi) \neq \emptyset$ and let $x \in L(k, \pi)$. Then $k(x) \subseteq \pi$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X, we have $k(1) \subseteq k(x) \subseteq \pi$. Thus, $1 \in L(k, \pi)$. Next, let $x, y \in X$ be such that $x, x \cdot y \in L(k, \pi)$. Then $k(x) \subseteq \pi$ and $k(x \cdot y) \subseteq \pi$. Since $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X, we have $k(y) \subseteq k(x \cdot y) \cup k(x) \subseteq \pi$. So $y \in L(k, \pi)$. Hence, $L(k, \pi)$ is a deductive system of X.

Conversely, assume that for all $\pi \in \mathcal{P}([0,1])$, the nonempty subsets $U(h, \pi)$ and $L(k, \pi)$ of X are deductive systems. Let $x \in X$. Then $h(x) \in \mathcal{P}([0,1])$. Choose $\pi = h(x) \in \mathcal{P}([0,1])$. Then $h(x) \supseteq \pi$. Thus, $x \in U(h, \pi) \neq \emptyset$. By assumption, we have $U(h, \pi)$ is a deductive system of X and thus $1 \in U(h, \pi)$. So $h(1) \supseteq \pi = h(x)$. Let $x, y \in X$. Then $h(x), h(x \cdot y) \in \mathcal{P}([0,1])$. Choose $\pi = h(x) \cap h(x \cdot y) \in \mathcal{P}([0,1])$. Then $h(x) \supseteq \pi$ and $h(x \cdot y) \supseteq \pi$. Thus, $x, x \cdot y \in U(h, \pi) \neq \emptyset$. By assumption, we have $U(h, \pi)$ is a deductive system of X and then $y \in U(h, \pi)$. Thus, $h(y) \supseteq \pi = h(x) \cap h(x \cdot y)$. Let $x \in X$. Then $k(x) \in \mathcal{P}([0,1])$. Choose $\pi_1 = k(x) \in \mathcal{P}([0,1])$. Then $k(x) \subseteq \pi_1$. Thus, $x \in L(k, \pi_1)$. By assumption, we have $L(k, \pi_1)$ is an ideal of X and thus $1 \in L(k, \pi_1) \neq \emptyset$. So $k(1) \subseteq \pi_1 = k(x)$. Let $x, y \in X$. Then $k(x), k(x \cdot y) \in \mathcal{P}([0,1])$. Choose $\pi_1 = k(x) \cup k(x \cdot y) \in \mathcal{P}([0,1])$. Then $k(x) \subseteq \pi_1$ and $k(x \cdot y) \subseteq \pi_1$. Thus, $x, x \cdot y \in L(k, \pi_1) \neq \emptyset$. By assumption, we have $L(k, \pi_1)$ is a deductive system of X and then $y \in L(k, \pi_1)$. Thus, $k(y) \subseteq \pi_1 = k(x) \cup k(x \cdot y)$. Hence, $\mathcal{H}$ is an intuitionistic hesitant fuzzy deductive system of X. □

Definition 3.6. *Let $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ be a family of IHFSs on a reference set X. We define the IHFS $\bigcap_{\alpha \in \Delta} \mathcal{H}_\alpha = \left(\bigcap_{\alpha \in \Delta} h_\alpha, \bigcup_{\alpha \in \Delta} k_\alpha \right)$ by $\left(\bigcap_{\alpha \in \Delta} h_\alpha \right)(x) = \bigcap_{\alpha \in \Delta} h_\alpha(x)$ and $\left(\bigcup_{\alpha \in \Delta} k_\alpha \right)(x) = \bigcup_{\alpha \in \Delta} k_\alpha(x)$ for all $x \in X$, which is called the intuitionistic hesitant intersection of IHFSs.*

Proposition 3.2. *If $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ is a family of intuitionistic hesitant fuzzy deductive systems of a Hilbert algebra $X = (X, \cdot, 1)$, then $\bigcap_{\alpha\in\Delta} \mathcal{H}_\alpha$ is an intuitionistic hesitant fuzzy deductive system of X.*

Proof: Let $\{\mathcal{H}_\alpha \mid \alpha \in \Delta\}$ be a family of intuitionistic hesitant fuzzy deductive systems of X. Let $x \in X$. Then $\left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(1) = \bigcap_{\alpha\in\Delta} h_\alpha(1) \supseteq \bigcap_{\alpha\in\Delta} h_\alpha(x) = \left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(x)$ and $\left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(1) = \bigcup_{\alpha\in\Delta} k_\alpha(1) \subseteq \bigcup_{\alpha\in\Delta} k_\alpha(x) = \left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(x)$. Let $x, y \in X$. Then $\left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(y) = \bigcap_{\alpha\in\Delta} h_\alpha(y) \supseteq \bigcap_{\alpha\in\Delta} (h_\alpha(x\cdot y) \cap h_\alpha(x)) = \left(\bigcap_{\alpha\in\Delta} h_\alpha(x\cdot y)\right) \cap \left(\bigcap_{\alpha\in\Delta} h_\alpha(x)\right) = \left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(x\cdot y) \cap \left(\bigcap_{\alpha\in\Delta} h_\alpha\right)(x)$ and $\left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(y) = \bigcup_{\alpha\in\Delta} k_\alpha(y) \subseteq \bigcup_{\alpha\in\Delta} (k_\alpha(x\cdot y) \cup k_\alpha(x)) = \bigcup_{\alpha\in\Delta} k_\alpha(x\cdot y) \cup \bigcup_{\alpha\in\Delta} k_\alpha(x) = \left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(x\cdot y) \cup \left(\bigcup_{\alpha\in\Delta} k_\alpha\right)(x)$. Hence, $\bigcap_{\alpha\in\Delta} \mathcal{H}_\alpha$ is an intuitionistic hesitant fuzzy deductive system of X. □

Definition 3.7. *Let $A = (h_A, k_A)$ and $B = (h_B, k_B)$ be IHFSs on sets X and Y, respectively. The Cartesian product $A \times B = (h, k)$ defined by $h(x, y) = h_A(x) \cap h_B(y)$ and $k(x, y) = k_A(x) \cup k_B(y)$, where $h : X \times Y \to \mathcal{P}([0, 1])$ and $k : X \times Y \to \mathcal{P}([0, 1])$ for all $x \in X$ and $y \in Y$.*

Remark 3.1. *Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. Then $(X \times Y, \diamond, (1_X, 1_Y))$ is a Hilbert algebra defined by $(x, y) \diamond (u, v) = (x \cdot u, y \star v)$ for every $x, u \in X$ and $y, v \in Y$.*

Proposition 3.3. *If $A = (h_A, k_A)$ and $B = (h_B, k_B)$ are two intuitionistic hesitant fuzzy deductive systems of Hilbert algebras X and Y, respectively, then the Cartesian product $A \times B$ is also an intuitionistic hesitant fuzzy deductive system of $X \times Y$.*

Proof: Let $(x, y) \in X \times Y$. Then $h(1_X, 1_Y) = h_A(1_X) \cap h_B(1_Y) \supseteq h_A(x) \cap h_B(y) = h(x, y)$ and $k(1_X, 1_Y) = k_A(1_X) \cup k_B(1_Y) \subseteq k_A(x) \cup k_B(y) = k(x, y)$. Let $(x_1, x_2), (y_1, y_2) \in X \times Y$. Then $h(y_1, y_2) = h_A(y_1) \cap h_B(y_2) \supseteq (h_A(x_1 \cdot y_1) \cap h_A(x_1)) \cap (h_B(x_2 \star y_2) \cap h_B(x_2)) = h_A(x_1 \cdot y_1) \cap h_B(x_2 \star y_2) \cap h_A(x_1) \cap h_B(x_2) = h(x_1 \cdot y_1, x_2 \star y_2) \cap h(x_1, x_2) = h((x_1, x_2) \diamond (y_1, y_2)) \cap h(x_1, x_2)$ and $k(y_1, y_2) = k_A(y_1) \cup k_B(y_2) \subseteq (k_A(x_1 \cdot y_1) \cup k_A(x_1)) \cup (k_B(x_2 \star y_2) \cup k_B(x_2)) = k_A(x_1 \cdot y_1) \cup k_B(x_2 \star y_2) \cup k_A(x_1) \cup k_B(x_2) = k(x_1 \cdot y_1, x_2 \star y_2) \cup k(x_1, x_2) = k((x_1, x_2) \diamond (y_1, y_2)) \cup k(x_1, x_2)$. Hence, $A \times B$ is an intuitionistic hesitant fuzzy deductive system of $X \times Y$. □

Lemma 3.3. *Two IHFSs $A = (h_A, k_A)$ and $B = (h_B, k_B)$ are intuitionistic hesitant fuzzy deductive systems of Hilbert algebras X and Y, respectively if and only if the IHFSs $\oplus(A \times B)$ and $\otimes(A \times B)$ are intuitionistic hesitant fuzzy deductive systems of $X \times Y$.*

Proof: It follows from Proposition 3.3 and Theorem 3.3. □

A mapping $f : (X, \cdot, 1_X) \to (Y, \star, 1_Y)$ of Hilbert algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \star f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras, then $f(1_X) = 1_Y$.

Definition 3.8. *Let f be a function from a nonempty set X to a nonempty set Y. If $\mathcal{H} = (h, k)$ is an IHFS on Y, then the IHFS $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f)$ in X is called the pre-image of $\mathcal{H}$ under f.*

Theorem 3.8. *Let $f : (X, \cdot, 1_X) \to (Y, \star, 1_Y)$ be a homomorphism of Hilbert algebras. If $\mathcal{H} = (h, k)$ is an intuitionistic hesitant fuzzy deductive system of Y, then $f^{-1}(\mathcal{H}) = (h \circ f, k \circ f)$ is an intuitionistic hesitant fuzzy deductive system of X.*

Proof: By assumption, $h(f(1_X)) = h(1_X) \supseteq h(y)$ for every $y \in Y$. In particular, $(h \circ f)(1_X) = h(f(1_X)) \supseteq h(f(x)) = (h \circ f)(x)$ for all $x \in X$. Also, $k(f(1_X)) = k(1_Y) \subseteq k(y)$ for every $y \in Y$. In particular, $(k \circ f)(1_X) = k(f(1_X)) \subseteq k(f(x)) = (k \circ f)(x)$ for all $x \in X$. Let $x, y \in X$. Then $(h \circ f)(y) = h(f(y)) \supseteq h(f(x) \star f(y)) \cap h(f(x)) = h(f(x \cdot y)) \cap h(f(x)) = (h \circ f)(x \cdot y) \cap (h \circ f)(x)$ and $(k \circ f)(y) = k(f(y)) \subseteq k(f(x) \star f(y)) \cup k(f(x)) = k(f(x \cdot y)) \cup k(f(x)) = (k \circ f)(x \cdot y) \cup (k \circ f)(x)$. Hence, $f^{-1}(\mathcal{H})$ is an intuitionistic hesitant fuzzy deductive system of X. $\square$

4. **Conclusion.** In the present paper, we have introduced the concept of intuitionistic hesitant fuzzy deductive systems of Hilbert algebras. Various interesting properties have been presented and the IHFSs $\oplus\mathcal{H}$ and $\otimes\mathcal{H}$ have also been studied. From Theorem 3.7, we have found a significant relationship between intuitionistic hesitant fuzzy deductive systems and their π-level subsets. Additionally, several related features are researched together with the homomorphic pre-images of intuitionistic hesitant fuzzy deductive systems of Hilbert algebras.

To extend the results of this paper, future research will focus on intuitionistic hesitant fuzzy sets in the concept of anti-type in Hilbert algebras. It can also be applied to other algebraic systems, and the results can be compared to those presented in this article.

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