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APPROXIMATION SOLUTION OF FUZZY VOLTERRA INTEGRAL EQUATION

Zahraa Abbas Ibrahim and Nabaa Najdi Hasan

Department of Mathematics
College of Science
Mustansiriyah University
PO Box 14022, Palestine Street, Baghdad, Iraq
zara.ali1431@gmail.com; alzaer1972@uomustansiriyah.edu.iq

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Abstract. *In this paper a study of numerical approximate solution to solve the following fuzzy Volterra integral equations (FVIE) is presented: 1) Linear fuzzy Volterra integral equation (LFVIE); 2) Non-linear fuzzy Volterra integral equation (NLFVIE); 3) Linear fuzzy Volterra integro differential equation (LINTGRO); 4) Non-linear fuzzy Volterra integro differential equation (NLINTGRO). Finally, linear non-polynomial spline algorithm is presented to solve some examples which implemented by using software MathCad15.*
Keywords: Fuzzy Volterra integral equation, Linear non-polynomial spline, Linear and non-linear fuzzy Volterra integro differential equation

1. **Introduction.** The concept of fuzzy numbers was originally introduced by Zadeh [1]. Fuzzy number was developed by Mizumoto and Tanaka [2], Nahmias [3], Dubois and Prade [4] for all fuzzy numbers as a location of α-levels, $0 \leq \alpha \leq 1$. Alternative approaches were later suggested by Goetschel and Voxman [5], Matloka [6] and others. The notions of fuzzy integral equation with fuzzy control have attracted researchers. Different definitions of fuzzy integrals have been established and extended to fuzzy calculus. The fuzzy integral equation theory played a key role in various fields of applied mathematics, physics, engineering, etc. Therefore, the study of fuzzy integral equations has been rapidly advancing in recent years [7-11]. In recent years, researches on fuzzy integral equation from both theoretical and numerical points of view have been developed. A numerical solution of some fuzzy integral equation using approximation methods, such as collocation method was produced [7]. [8] proposed a technique to remove the singular behavior of Lane-Emden equation and provide the high precision approximation solution by cubic non-polynomial spline method. Among the approximation methods for solving fuzzy Volterra integral equations (FVIE) in [9] the homotopy analysis method is semi analytic to solve the linear and non-linear fuzzy Volterra integral equation. Recently, Padmapriya et al. [10] investigated the numerical solution of fuzzy fractional delay differential equation using the proposed novel technique. [11] studied the homotopy perturbation Sumudu transform method is employed to find the analytical solution of non-linear integro differential equation. Finally in [12] Hasan and Nasif used cubic trigonometric spline for solving non-linear Volterra integral equation. In this paper, we drive linear non-polynomial spline to solve main classes of FVIE (linear, non-linear and integrodifferential). The paper is organized as follows: in Section 2 preliminaries of fuzzy sets are given, in Section 3 parametric form of FVIE is described, in Section 4 linear non-polynomial spline functions are derived, in Section 5 illustrated examples show the accuracy of the method and finally conclusion is given in Section 6.

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2. **Preliminaries.** In this section, basic notations used in the fuzzy operations are introduced [13].

Definition 2.1. *If X is a collection of objects denoted by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs denoted and defined by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))/x \in X\}$, where $\mu_{\tilde{A}}(x)$ is called membership function or grade of membership (also degree of compatibility or degree of truth) of x in $\tilde{A}$ which maps X to $[0, 1]$.*

Definition 2.2. *α-cut of a fuzzy $\tilde{A}$ set is a crisp set A_α and defined by A_α or $\tilde{A}[\alpha] = \{x/\mu_{\tilde{A}}(x) \geq \alpha\}$, where $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}$.*

Definition 2.3. *A fuzzy set $\tilde{A}$ is said to be convex fuzzy set if A_α is a convex set for all $\alpha \in (0, 1]$.*

Definition 2.4. *A fuzzy set $\tilde{A}$ is said to be normal fuzzy set if there exists an element $(\alpha, 1) \in \tilde{A}$.*

Definition 2.5. *If a fuzzy set is convex, normalized and its membership function, defined in $\mathbb{R}$, is piecewise continuous, then it is called as fuzzy number. A triangular fuzzy number $\tilde{A}$ is denoted by (a_1, a_2, a_3) and it is a fuzzy set $(x, \mu_{\tilde{A}}(x))$ where*

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3 \\ 0, & otherwise \end{cases}$$

$\tilde{A}$ is called positive triangular fuzzy number if $a_1 > 0$ and negative triangular fuzzy number if $a_3 < 0$.

3. **Parametric Form of FVIE.** The second kind VIE [9] is defined below:

$$u(x) = f(x) + \lambda \int_0^X k(x, t, u(t))dt \tag{1}$$

where λ is a positive fuzzy parameter, k is an arbitrary function defined over $[a, b] \times [a, b]$ where $a \leq t$, $x \leq b$, $t > x$ and $f(x)$ is a given function of $x \in [a, b]$ with $u(x)$ as the unknown function to be determined. The fuzzy VIE of the second kind according to [9]:

$$u(x, r) = f(x, r) + \lambda \int_0^x k(x, t, u(t, r); r)dt \tag{2}$$

where

$$\left.\begin{array}{l} u(x) = u(x, r) = [\underline{u}(x, r), \bar{u}(x, r)] \\ f(x) = f(x, r) = \left[\underline{f}(x, r), \bar{f}(x, r)\right] \\ k(x, t, u(t)) = k(x, t, u(t); r) = \left[\underline{k}\left(x, t, \underline{u}(t, r); r\right), \bar{k}\left(x, t, \bar{u}(t, r); r\right)\right] \\ u(t) = u(t, r) = [\underline{u}(t, r), \bar{u}(t, r)] \end{array}\right\} \tag{3}$$

here $0 \leq r \leq 1$, then solution of Equation (1) can be obtained by solving the following two integral equations:

$$\left.\begin{array}{l} \underline{u}(x, r) = \underline{f}(x, r) + \lambda \displaystyle\int_0^x \underline{k}\big(x, t, \underline{u}(t, r); r\big)dt \\ \bar{u}(x, r) = \bar{f}(x, r) + \lambda \displaystyle\int_0^x \bar{k}\big(x, t, \bar{u}(t, r); r\big)dt \end{array}\right\} \tag{4}$$

4. **Linear Non-Polynomial Splines [14].** Let $\Delta = \{r_0, r_1, r_2, \ldots, r_n\}$ be partition of interval $[a, b]$, and $S(\Delta)$ denote the set of piecewise polynomials on subinterval $I_i = [r_i, r_{i+1}]$ of partition Δ. We consider linear non-polynomial spline method for finding approximation solution of fuzzy Volterra integral equation (FVIE) of the second kind considering the grid point r_i on the interval $[a, b]$ as follows:

$$a = r_0 < r_1 < r_2 < \cdots < r_n \tag{5}$$

$$r_i = r_0 + ih, \quad i = 0, 1, 2, \ldots, n \tag{6}$$

$$h = \frac{b-a}{n} \tag{7}$$

where n is appositive integer. Let $u(r)$ be the exact solution of Equation (2) and $S_i(r)$ be an approximation to $u_i = u(r_i)$ obtained by the segment $S_i(r)$, each non-polynomial spline segment $S_i(r)$ has the form

$$S_i(r) = a_i \sin k(r - r_i) + b_i \cos k(r - r_i) + c_i(r - r_i) + d_i \tag{8}$$

where a_i, b_i, c_i and d_i are constant and k is the frequency of the trigonometric function which will be used to raise the accuracy of method. We consider the following relations:

$$\begin{aligned}
S_i(r_i) &= u(r_i) \\
S_i'(r_i) &= kb_i + c_i \approx u_i'(r_i) \\
S_i''(r_i) &= -k^2 b_i \approx u_i''(r_i) \\
S_i'''(r_i) &= -k^3 a_i \approx u_i'''(r_i)
\end{aligned}$$

Now, we can obtain the values of a_i, b_i, c_i and d_i as follows:

$$a_i = \frac{-1}{k^3} u'''(r_i) \tag{9}$$

$$b_i = \frac{-1}{k^2} u''(r_i) \tag{10}$$

$$c_i = u'(r_i) + \frac{-1}{k^3} u'''(r_i) \tag{11}$$

$$d_i = u(r_i) + \frac{1}{k} u''(r_i) \tag{12}$$

for $i = 0, 1, 2, \ldots, n$.

The method.

Consider the (FVIE) of the second kind to solve we differentiate Equation (2) three times with respect to r then put $r = a$ to get [12]:

$$u_0 = u(a) = f(a) \tag{13}$$

$$u_0' = u'(a) = f'(a) + k(a, a)u(a) \tag{14}$$

Let $E(x, t, u(t, r); r) = \dfrac{\partial k(x, t, u(t, r))}{\partial r}$

$$u''(r) = f_0''(r) + \int_0^x \frac{\partial E(x, t, u(t, r); r)}{\partial r} dt + 2E(x, t, u(t, r); r)$$

$$u_0''(a) = u''(a) = f_0''(a) + 2E(a, a, u(a)) \tag{15}$$

Let $F(x, t, u(t, r); r) = \dfrac{\partial E(x, t, u(t, r); r)}{\partial r}$

$$u'''(r) = f'''(r) + \int_0^x \frac{\partial F(x, t, u(t, r); r)}{\partial r} dt + 3F(x, t, u(t, r); r)$$

$$u_0'''(a) = f'''(a) + 3F(a, a, a, u(a)) \tag{16}$$

Thus, we can approximate the solution of (FVIE) of second kind as the following algorithm.

Algorithm

Step1: Input $h = (b-a)/n$, $r_i = r_0 + ih$, $i = 0, 1, \ldots, n$.

Step2: Compute a_i, b_i, c_i and d_i by substituting Equations (13)-(16) in Equations (9)-(12).

Step3: Evaluate $S_0(r)$ by using Step2 and Equation (8) for $i = 0$.

Step4: Approximate $u_1 = u(r_1) \approx S_0(r_1)s$.

Step5: Do the following steps for $i = 1$ to $n-1$.

Step6: Compute a_i, b_i, c_i and d_i by using Equations (9)-(12) and replacing $u_0(r_i)$, $u_0''(r_i)$ and $u_0'''(r_i)$ in $S(r_i)$, $S''(r_i)$ and $S'''(r_i)$.

Step7: Approximate $u_{i+1} = S_i(r_{i+1})$.

5. **Illustrative Examples.** In this section, four test examples are illustrated below to solve main classes of FVIE (linear, non-linear, integrodifferential), $s(x,r)$ the approximate solution by the proposed method and error $= |u(x,r) - s(x,r)|$ where $u(x,r)$ the exact solution.

Example (1): Consider LFVIE

$u(x,r) = f(x,r) + \int_0^x xtu(r,t)dt$, where $f(x,r) = \left[x^3 - \frac{x^6}{5}\left(r^2+r\right); x^3 - \frac{x^6}{5}\left(4 - r^3 - r\right)\right]$. The exact solution is $u(x,r) = \left[x^3\left(r^2+r\right); x^3\left(4-r^3-r\right)\right]$ [7].

TABLE 1. Results of Example (1)

r	$\underline{u}(x,r)$	$\underline{s}(x,r)$	Error	$\bar{u}(x,r)$	$\bar{s}(x,r)$	Error
0	0.0040000	0.0039992	8.0×10^{-7}	0.0000000	0.0000000	0.0000000
1/3	0.0036962	0.0036291	5.9×10^{-7}	0.0004440	0.0004433	1.1×10^{-6}
2/3	0.0030370	0.0030429	5.9×10^{-5}	0.0011111	0.0010946	1.6×10^{-5}
1	0.0020000	0.0020484	4.8×10^{-5}	0.0020000	0.0019190	8.0×10^{-5}

Example (2): Consider LINTGRO

$y'(x,r) = g(x,r) + \int_0^x y(t,r)dt$, where $g(x,r) = [(r-1)x, (1-r)x]$, $y(x,0) = -x$, $y'(x,0) = x$. The exact solution $y(x,r) = [(r-1)\sinh(x), (1-r)\sinh(x)]$ [9].

TABLE 2. Results of Example (2)

r	$\underline{u}(x,r)$	$\underline{s}(x,r)$	Error	$\bar{u}(x,r)$	$\bar{s}(x,r)$	Error
0	-0.0100166	-0.1000000	1.66×10^{-4}	0.10016675	0.1000000	1.66×10^{-4}
0.2	-0.0801334	-0.0800000	1.33×10^{-4}	0.08013340	0.0800000	1.33×10^{-4}
0.4	-0.0601167	-0.0600000	1.00×10^{-4}	0.06011672	0.0600000	1.00×10^{-4}
0.6	-0.0400667	-0.0400000	6.67×10^{-5}	0.04006670	0.0400000	6.67×10^{-5}
0.8	-0.0200333	-0.0200000	3.33×10^{-5}	0.02003335	0.0200000	3.33×10^{-5}
1	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000

Example (3): Consider NLFVIE

$u(x,r) = f(x,r) + \int_0^x x^2(1-2t)u^2(t,r)dt$, where $f(x,r) = \left[(2-r)^2\left(\frac{x^6}{2} + x^5 + x^3 + \frac{11x^2}{32}\right) - \frac{11x^2}{32} + rx + r, r^2\left(\frac{x^6}{2} + x^5 + x^3 + \frac{11x^2}{32}\right) + (2-r)\left(\frac{-11}{32}(2-r)x^2 + x + 1\right)\right]$. The exact solution $u(x,r) = [r(x+1); (2-r)(x+1)]$ [15].

TABLE 3. Results of Example (3)

r	$\underline{u}(x,r)$	$\underline{s}(x,r)$	Error	$\bar{u}(x,r)$	$\bar{s}(x,r)$	Error
0	0.0000000	0.0063545	6.354×10^{-3}	2.2000000	2.1860000	0.0140000
0.2	0.2200000	0.2243961	4.396×10^{-3}	1.9800000	1.9640000	0.0160000
0.4	0.4400000	0.4424377	2.438×10^{-3}	1.7600000	1.7410000	0.0190000
0.6	0.6600000	0.6604793	4.793×10^{-4}	1.5400000	1.5190000	0.0210000
0.8	0.8800000	0.8785209	1.479×10^{-3}	1.3200000	1.2970000	0.0230000
1	1.1000000	1.0965625	3.438×10^{-3}	1.1000000	1.0760000	0.0240000

Example (4): Consider NLINTGRO

$y'(x,r) = g(x,r) + \int_0^x y^2(x,r)dt$, where

$$\underline{g}(x,r) = \left[\left(r^2+r\right)x^2 - \frac{\left(r^4+2r^3+r^2\right)x^6+\left(r^2+r\right)^2x^6}{30} + \frac{\left(r^4+2r^3+r^2\right)^2x^{14}}{163800} - \frac{\left(r^2+r\right)\left(r^4+2r^3+r^2\right)x^{10}}{1350}\right]$$

and

$$\bar{g}(x,r) = \left[\left(4-r^3+r\right)x^2 - \frac{\left(16-8r^3-8r+r^6+2r^4+r^2\right)x^6+\left(4-r^3+r\right)^2x^6}{30} + \frac{\left(16-8r^3-8r+r^6+2r^4+r^2\right)^2x^{14}}{163800} - \frac{\left(4-r^3+r\right)\left(16-8r^3-8r+r^6+2r^4+r^2\right)x^{10}}{1350}\right].$$

$$y_0(x,0) = \left[\frac{-x^{10}}{30} + \frac{x^2}{163800} - \frac{x^{14}}{1350}; 4x^2 + \frac{16x^{14}}{163800} - \frac{64x^{10}}{1350}\right];$$

$$y_0'(x,0) = \left[x^2 - \frac{x^{14}}{1350}; x^2 + \frac{16x^6}{30} + \frac{128x^{14}}{81900} + \frac{16x^{10}}{1350}\right].$$

The exact solution $y(x,r) = \left[\left(r^2+r\right)x^2, \left(4-r^3+r\right)x^2\right]$ [11].

TABLE 4. Results of Example (4)

r	$\underline{u}(x,r)$	$\underline{s}(x,r)$	Error	$\bar{u}(x,r)$	$\bar{s}(x,r)$	Error
0	0.0000000	-3.33×10^{-12}	3.333×10^{-12}	0.0000000	3.33×10^{-12}	4.74×10^{-12}
0.2	2.4×10^{-3}	2.417×10^{-3}	1.728×10^{-5}	0.0024000	0.0024172	2.66×10^{-7}
0.4	5.6×10^{-3}	5.674×10^{-3}	7.362×10^{-5}	0.0056000	0.0056737	5.31×10^{-6}
0.6	9.6×10^{-3}	9.751×10^{-3}	1.511×10^{-4}	0.0096000	0.0097511	3.88×10^{-5}
0.8	0.0140000	0.0150000	1.986×10^{-4}	0.0144000	0.0145985	1.61×10^{-4}
1	0.0200000	0.0200000	0.0000000	0.0200000	0.0201344	4.88×10^{-4}

In Figures 1, 2, 3, and 4, we plot the graphs of Examples (1), (2), (3), (4) of the approximate and exact solutions for fuzzy parameter r, $0 \le r \le 1$, where s(r) approximate, e(r) exact for upper solutions and s1(r) approximate, e1(r) exact for lower solutions.

In above examples, we use $r = 0, 0.1, \ldots, 1$, where we calculate the error of the exact solution and obtained solution of fuzzy Volterra integral equation with linear non-polynomial spline. The tables show the convergence behavior of the method, the exact and obtained solution of fuzzy Volterra integral equation at $x = 0.1$ and for $r = 0, 0.1, \ldots, 1$ are shown in figures.

6. **Conclusion.** In this work, we have considered the fuzzy Volterra integral equation (FVIE) with parametric form. The linear non-polynomial spline has been successfully used to obtain the approximate solutions; the trigonometric term of this spline has infinite derivative which gets good agreement with exact solution. We derived an algorithm to solve main classes of FVIE (linear, non-linear and integrodifferential) numerically. The examples show that the results of the method are convergent to the exact solution. In

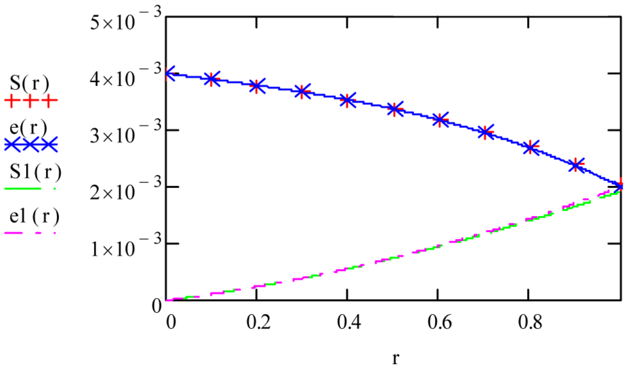

FIGURE 1. Results of Example (1)

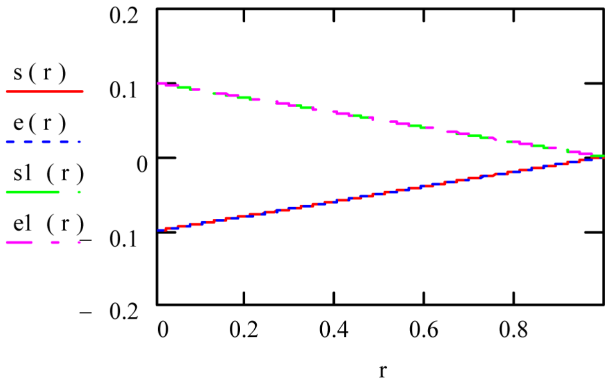

FIGURE 2. Results of Example (2)

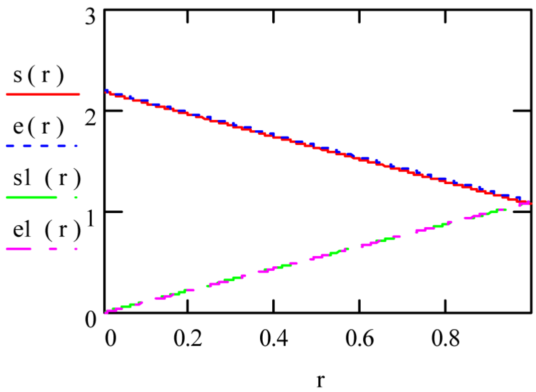

FIGURE 3. Results of Example (3)

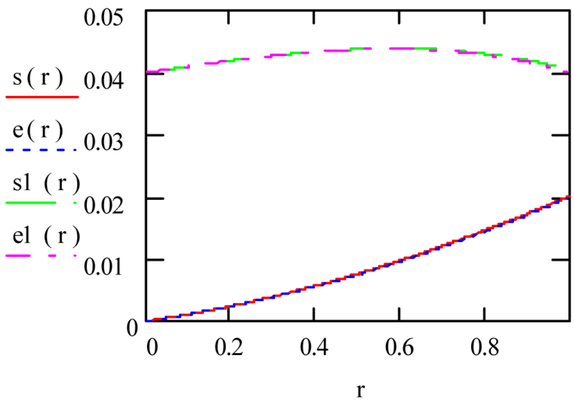

FIGURE 4. Results of Example (4)

view of the results, tables and figures show that the proposed technique is a powerful mathematical tool for solving FVIE with MathCad15 programing implementation. Also, we can develop the method to obtain approximate solution of fuzzy integral equation of fractional order.

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