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M-BI-BASE GENERATOR OF ORDERED Γ-SEMIGROUPS

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Abstract. *In this paper, we introduce the notion of M-bi-bases of an ordered* Γ*-semigroup, which is a generalization of the bi-base based on a* Γ*-semigroup and an ordered semigroup. Some of their characterizations are obtained through M-bi-bases. Let* $\mathbb{W}$ *be an M-bi-base of an ordered* Γ*-semigroup* $\mathbb{S}$ *and* $z_1, z_2, z_3 \in \mathbb{W}$*. If* $z_1 \in (N(z_3 \times \Gamma \times z_2) \cup z_3 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2]$*, then prove that either* $z_1 = z_2$ *or* $z_1 = z_3$*. If* $\mathbb{W}$ *is an M-bi-base of* $\mathbb{S}$ *and* $z_1, z_2 \in \mathbb{W}$ *and* $z_1 \neq z_2$*, then show that neither* $z_1 \leq_{mb} z_2$*, nor* $z_2 \leq_{mb} z_1$*. In addition, we discuss an M-bi-base, which is generated by an element and a subset, and we introduce the concept of a quasi order, which is based on an M-bi-base. With the help of some examples those are discussed.*

Keywords: Ordered Γ-semigroup, M-bi-ideal, M-bi-base, Quasi order

1. **Introduction.** Several authors and researchers have characterized the many different ideals based on Γ-semigroup [1], and Γ-semiring [2]. A partial order is a relation "≤" which satisfies the conditions such as reflexivity, antisymmetry, and transitivity. The different classes of semigroup and Γ-semigroup have been characterized based on bi-ideal [3, 4, 5]. The origin of an ordered semigroup as a generalization of an ordinary semigroup with a partially ordered relation is constructed on a semigroup such that the relation is compatible with the operation. Sen and Seth have discussed an ordered Γ-semigroup [6] and it has been studied by several authors [7, 8, 9, 10]. The notion of a bi-ideal of semirings and semigroups is a generalization of the notion of an ideal of semirings and semigroups. An ordered Γ-semigroup is a generalization of Γ-semigroups. As a result, the notion of an ordered bi-ideal of an ordered semigroup is a generalization of the notion of a bi-ideal of semigroups. The notion of bi-ideals in semigroups was introduced by Lajos [11]. The concept of a bi-ideal is a very interesting and important thing in semiring. The bi-ideal is a generalization of the left and right ideals. Many mathematicians proved important results and characterizations of algebraic structures by using various ideals. An M-bi-ideal of a semigroup is a generalization of the concept of a bi-ideal of a semigroup [12]. In the same way, the notion of an ordered M-bi-ideal of an ordered semigroup is a generalization of the ordered bi-ideal of an ordered semigroup. From a pure algebraic point of view,

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the important properties of M-bi-base have been described. Because of these motivating facts, it is natural to generalize semigroup results to Γ-semigroups and Γ-semigroups to ordered Γ-semigroups. Jantanan et al. [13] discussed the bi-base of an ordered Γ-semigroup. Palanikumar and Arulmozhi discussed various ideals based on semirings and ternary semirings [14, 15, 16]. Recently, Sanpan et al. [17] discussed the new logical theory for the regularities of ordered gamma semihypergroups.

Our purpose in this paper is to examine many important results of M-bi-base of ordered Γ-semigroups and then to characterize them through M-bi-ideal and M-bi-base. Furthermore, we show how the element and subset of an ordered Γ-semigroup generate the M-bi-ideal and M-bi-base. The paper is organized into four sections as follows. Section 1 is called an introduction. In Section 2, a brief description of the ordered Γ-semigroup information is given. Section 3 provided a numerical example of the M-base generator. Finally, a conclusion is provided in Section 4. The purpose of this paper is

1) To show how to generate M-bi-ideals from an ordered Γ-semigroup;
2) The relationship "$\leq$" based on M-bi-base is not a partial order;
3) To illustrate, the subset of M-bi-base is not an M-bi-base.

2. **Background.**

We present a brief summary of the basic notions and concepts used in an ordered Γ-semigroup that will be of high value for our later pursuits. In this article, $\mathbb{S}$ denotes an ordered Γ-semigroup, unless otherwise stated.

Definition 2.1. [1] *Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup if there exists a function from $S \times \Gamma \times S \to S$ which maps $(z_1, \xi, z_2) \to z_1\xi z_2$ satisfying the condition $(z_1\xi z_2)\nu z_3 = z_1\xi(z_2\nu z_3)$ for all $z_1, z_2, z_3 \in S$ and $\xi, \nu \in \Gamma$.*

Definition 2.2. [6] *The algebraic system $(\mathbb{S}, \Gamma, \leq)$ is said to be an ordered Γ-semigroup if it satisfies the following conditions:*

(1) $\mathbb{S}$ is a Γ-semigroup,
(2) $\mathbb{S}$ is a partially ordered set (poset) elicited from "$\leq$",
(3) If $s'' \leq s'''$, then $s''\xi s' \leq s'''\xi s'$ and $s'\xi s'' \leq s'\xi s'''$, for any $s', s'', s''' \in \mathbb{S}$ and $\xi \in \Gamma$.

Definition 2.3. *Let $\mathbb{W} \subseteq \mathbb{S}$ be called an ordered Γ-bi-ideal if it satisfies the following conditions:*

(1) $\mathbb{W}$ is a Γ-subsemigroup,
(2) $\mathbb{W}\Gamma\mathbb{S}\Gamma\mathbb{W} \subseteq \mathbb{W}$,
(3) If $w \in \mathbb{W}$, and $s' \in \mathbb{S}$, such that $s' \leq w$, then $s' \in \mathbb{W}$.

Remark 2.1. [6] *For $X', X'' \subseteq \mathbb{S}$,*

(1) $X'\Gamma X'' = \{x'\xi x'' | x' \in X', x'' \in X'', \xi \in \Gamma\}$,
(2) $(X'] = \{s \in \mathbb{S} | s \leq x' \text{ for some } x' \in X'\}$,
(3) $X' \subseteq (X']$,
(4) If $X' \subseteq X''$, then $(X'] \subseteq (X'']$ and $(X']\Gamma(X''] \subseteq (X'\Gamma X'']$.

Lemma 2.1. *For $\mathbb{W} \subseteq \mathbb{S}$ and $a \in \mathbb{S}$,*

(1) $(\mathbb{W} \cup \mathbb{W}\Gamma\mathbb{W} \cup \mathbb{W}\Gamma\mathbb{S}\Gamma\mathbb{W}]$ is a smallest Γ-bi-ideal of $\mathbb{S}$ containing $\mathbb{W}$.
(2) $\langle a\rangle_b = (a \cup a\Gamma a \cup a\Gamma\mathbb{S}\Gamma a]$ is a smallest Γ-bi-ideal of $\mathbb{S}$ containing "a".

Definition 2.4. [13] *Let $\mathbb{W} \subseteq \mathbb{S}$ be called a bi-base of $\mathbb{S}$ if it satisfies the following conditions:*

(1) $\mathbb{S} = \langle\mathbb{W}\rangle_b$,
(2) If $\mathbb{V} \subseteq \mathbb{W}$ such that $\mathbb{S} = \langle\mathbb{V}\rangle_b$, then $\mathbb{V} = \mathbb{W}$.

3. ***M*-bi-base Generator.** We communicate some results on M-bi-ideal and its generator.

Definition 3.1. *Let $\mathbb{S}$ be an ordered Γ-semigroup, $\mathbb{W} \subseteq \mathbb{S}$ is called an M-bi-ideal of $\mathbb{S}$ if it satisfies the following conditions:*

(1) $\mathbb{W}$ is a Γ-subsemigroup,
(2) $\mathbb{W} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}$ M-times$) \times \Gamma \times \mathbb{W} \subseteq \mathbb{W}$,
(3) If $w \in \mathbb{W}$ and $s \in \mathbb{S}$ such that $s \leq w$, then $s \in \mathbb{W}$.

Remark 3.1. *For $z_1 \in \mathbb{S}$ and N, M are positive integers, then the following statements hold:*

(1) $Nz_1 = z_1 \times \Gamma \times z_1 \times \Gamma \times \cdots \times \Gamma \times z_1$ (N-times),
(2) $\mathbb{S} \times \Gamma \times \mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}$ (M-times) $\subseteq \mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}$ (M-1 times).

Theorem 3.1.

(1) For $z_1 \in \mathbb{S}$, the M-bi-ideal generated by "a" is $\langle z_1 \rangle_{mb} = \{z_1 \cup N(z_1 \times \Gamma \times z_1) \cup z_1 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}$ M-times$) \times \Gamma \times z_1\}$ and $N \geq M$, where N and M are positive integers,
(2) For $\mathbb{W} \subseteq \mathbb{S}$, the M-bi-ideal generated by "$\mathbb{W}$" is $\langle \mathbb{W} \rangle_{mb} = \{\mathbb{W} \cup \mathbb{W} \times \Gamma \times \mathbb{W} \cup \mathbb{W} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}$ M-times$) \times \Gamma \times \mathbb{W}\}$.

Definition 3.2. *Let $\mathbb{W} \subseteq \mathbb{S}$ be called an M-bi-base of $\mathbb{S}$ if it satisfies the following conditions:*

(1) $\mathbb{S} = \langle \mathbb{W} \rangle_{mb}$,
(2) If $\mathbb{V} \subseteq \mathbb{W}$ such that $\mathbb{S} = \langle \mathbb{V} \rangle_{mb}$, then $\mathbb{V} = \mathbb{W}$.

Example 3.1. *Let $\mathbb{S} = \{o_1, o_2, o_3, o_4, o_5, o_6\}$ and $\Gamma = \{\xi_1, \xi_2\}$, where ξ_1, ξ_2 are defined on $\mathbb{S}$ with the following table.*

ξ_1	o_1	o_2	o_3	o_4	o_5	o_6
o_1	o_1	o_1	o_1	o_1	o_1	o_6
o_2	o_1	o_1	o_1	o_2	o_3	o_6
o_3	o_1	o_2	o_3	o_1	o_1	o_6
o_4	o_1	o_1	o_1	o_4	o_5	o_6
o_5	o_1	o_4	o_5	o_1	o_1	o_6
o_6	o_6	o_6	o_6	o_6	o_6	o_6

ξ_2	o_1	o_2	o_3	o_4	o_5	o_6
o_1	o_1	o_4	o_1	o_4	o_4	o_6
o_2	o_1	o_2	o_1	o_4	o_4	o_6
o_3	o_1	o_4	o_3	o_4	o_5	o_6
o_4	o_1	o_4	o_1	o_4	o_4	o_6
o_5	o_1	o_4	o_3	o_4	o_5	o_6
o_6	o_6	o_6	o_6	o_6	o_6	o_6

$\leq := \{(o_1, o_1), (o_1, o_6), (o_2, o_2), (o_2, o_6), (o_3, o_3), (o_3, o_6), (o_4, o_4), (o_4, o_6), (o_5, o_5), (o_5, o_6), (o_6, o_6)\}$. *Clearly, $(\mathbb{S}, \Gamma, \leq)$ is an ordered Γ-semigroup. The covering relation $\leq := \{(o_1, o_6), (o_2, o_6), (o_3, o_6), (o_4, o_6), (o_5, o_6)\}$ is represented by Figure 1. Here, $\mathbb{W} = \{o_4, o_5\}$ is an M-bi-base of $\mathbb{S}$. The set of all non-empty proper subsets of $\mathbb{W}$ is not an M-base of $\mathbb{S}$.*

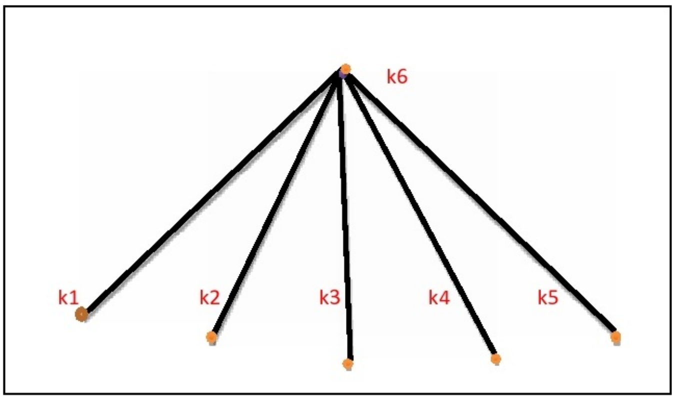

FIGURE 1. Covering relation

Theorem 3.2. *Let $\mathbb{W}$ be an M-bi-base of $\mathbb{S}$ and $z_1, z_2 \in \mathbb{W}$. If $z_1 \in (N(z_2 \times \Gamma \times z_2) \cup z_2 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2]$, then $z_1 = z_2$.*

Proof: Assume that $z_1 \in (N(z_2 \times \Gamma \times z_2) \cup z_2 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2]$, and suppose that $z_1 \neq z_2$. Let $\mathbb{V} = \mathbb{W} \setminus \{z_1\}$. Obviously, $\mathbb{V} \subset \mathbb{W}$. Since $z_1 \neq z_2$, $z_2 \in \mathbb{V}$. To show that $\langle \mathbb{V} \rangle_{mb} = \mathbb{S}$, clearly, $\langle \mathbb{V} \rangle_{mb} \subseteq \mathbb{S}$. It remains to prove that $\mathbb{S} \subseteq \langle \mathbb{V} \rangle_{mb}$. Let $s \in \mathbb{S}$. By hypothesis, $\langle \mathbb{W} \rangle_{mb} = \mathbb{S}$ and hence $s \in (\mathbb{W} \cup \mathbb{W} \times \Gamma \times \mathbb{W} \cup \mathbb{W} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{W}]$. We have $s \leq w$ for some $w \in \mathbb{W} \cup \mathbb{W} \times \Gamma \times \mathbb{W} \cup \mathbb{W} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{W}$. We can observe the following cases.
Case-1: Let $w \in \mathbb{W}$. There are two subcases to survey.
Subcase-1(a): Let $w \neq z_1$, then $w \in \mathbb{W} \setminus \{z_1\} = \mathbb{V} \subseteq \langle \mathbb{V} \rangle_{mb}$.
Subcase-1(b): Let $w = z_1$. We have $w = z_1 \in (N(z_2 \times \Gamma \times z_2) \cup z_2 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2] \subseteq (\mathbb{V} \times \Gamma \times \mathbb{V} \cup \mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}] \subseteq \langle \mathbb{V} \rangle_{mb}$.
Case-2: Let $w \in \mathbb{W} \times \Gamma \times \mathbb{W}$. Then $w = w_1 \times \xi \times w_2$, for some $w_1, w_2 \in \mathbb{W}$ and $\xi \in \Gamma$. Then there are four subcases to regard.
Subcase-2(a): Let $w_1 = z_1$ and $w_2 = z_1$. Now, $w = w_1 \times \xi \times w_2 = z_1 \times \xi \times z_1 \subseteq ((N(z_2 \times \Gamma \times z_2) \cup z_2 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2) \times \Gamma \times (N(z_2 \times \Gamma \times z_2) \cup z_2 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2)] \subseteq (\mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}] \subseteq \langle \mathbb{V} \rangle_{mb}$.
Subcase-2(b): Let $w_1 \neq z_1$ and $w_2 = z_1$. Now, $w = w_1 \times \xi \times w_2 \subseteq ((\mathbb{W} \setminus \{z_1\}) \times \Gamma \times (N(z_2 \times \Gamma \times z_2) \cup z_2 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2)] \subseteq (\mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}] \subseteq \langle \mathbb{V} \rangle_{mb}$.
Subcase-2(c): Let $w_1 = z_1$ and $w_2 \neq z_1$. Now, $w = w_1 \times \xi \times w_2 \subseteq ((N(z_2 \times \Gamma \times z_2) \cup z_2 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2) \times \Gamma \times (\mathbb{W} \setminus \{z_1\})] \subseteq (\mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}] \subseteq \langle \mathbb{V} \rangle_{mb}$.
Subcase-2(d): Let $w_1 \neq z_1$ and $w_2 \neq z_1$ and $\mathbb{V} = \mathbb{W} \setminus \{z_1\}$. Now, $w = w_1 \times \xi \times w_2 \in (\mathbb{W} \setminus \{z_1\}) \times \Gamma \times (\mathbb{W} \setminus \{z_1\}) = \mathbb{V} \times \Gamma \times \mathbb{V} \subseteq \langle \mathbb{V} \rangle_{mb}$.
Case-3: Let $w \in \mathbb{W} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{W}$. Then $w = w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4$ for some $w_3, w_4 \in \mathbb{W}$, $s_1, s_2, \ldots, s_n \in \mathbb{S}$ and $\xi, \nu, \xi_1, \xi_2, \ldots, \xi_n \in \Gamma$. There are four subcases to examine.
Subcase-3(a): Let $w_3 = z_1$ and $w_4 = z_1$. Now, $w = w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 = z_1 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_1 \subseteq ((N(z_2 \times \Gamma \times z_2) \cup z_2 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2) \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times (N(z_2 \times \Gamma \times z_2) \cup z_2 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2)] \subseteq (\mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}] \subseteq \langle \mathbb{V} \rangle_{mb}$.
Subcase-3(b): Let $w_3 \neq z_1$ and $w_4 = z_1$. Now, $w = w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \subseteq ((\mathbb{W} \setminus \{z_1\}) \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times (N(z_2 \times \Gamma \times z_2) \cup z_2 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2)] \subseteq (\mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}] \subseteq \langle \mathbb{V} \rangle_{mb}$.
Subcase-3(c): Let $w_3 = z_1$ and $w_4 \neq z_1$. Now, $w = w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \subseteq ((N(z_2 \times \Gamma \times z_2) \cup z_2 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2) \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times (\mathbb{W} \setminus \{z_1\})] \subseteq (\mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}] \subseteq \langle \mathbb{V} \rangle_{mb}$.
Subcase-3(d): Let $w_3 \neq z_1$ and $w_4 \neq z_1$ and $\mathbb{V} = \mathbb{W} \setminus \{z_1\}$. Now, $w = w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \subseteq (\mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}] \subseteq \langle \mathbb{V} \rangle_{mb}$. For all the cases, we have $\mathbb{S} \subseteq \langle \mathbb{V} \rangle_{mb}$. Thus, $\mathbb{S} = \langle \mathbb{V} \rangle_{mb}$. It is a contradiction, hence $z_1 = z_2$.

Theorem 3.3. *Let $\mathbb{W}$ be an M-bi-base of $\mathbb{S}$ and $z_1, z_2, z_3 \in \mathbb{W}$. If $z_1 \in (N(z_3 \times \Gamma \times z_2) \cup z_3 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2]$, then $z_1 = z_2$ or $z_1 = z_3$.*

Proof: Proof follows from Theorem 3.2.

Definition 3.3. *For any $s_1, s_2 \in \mathbb{S}$, $s_1 \leq_{mb} s_2 \Longleftrightarrow \langle s_1 \rangle_{mb} \subseteq \langle s_2 \rangle_{mb}$ is called a quasi order on $\mathbb{S}$.*

Remark 3.2. *The order $\leq_{mb}$ is not a partial order of $\mathbb{S}$.*

Example 3.2. *By Example 3.1, Clearly, $\langle o_2 \rangle_{mb} \subseteq \langle o_6 \rangle_{mb}$ and $\langle o_6 \rangle_{mb} \subseteq \langle o_2 \rangle_{mb}$ but $o_2 \neq o_6$. Hence, $\leq_{mb}$ is not a partial order on $\mathbb{S}$.*

If $\mathbb{V}$ is an M-bi-base of $\mathbb{S}$, then $\langle\mathbb{V}\rangle_{mb} = \mathbb{S}$. Let $s \in \mathbb{S}$. Then $s \in \langle\mathbb{V}\rangle_{mb}$ and so $s \in \langle z_1\rangle_{mb}$ for some $z_1 \in \mathbb{V}$. This implies $\langle s\rangle_{mb} \subseteq \langle z_1\rangle_{mb}$. Hence, $s \leq_{mb} z_1$.

Remark 3.3. *If $\mathbb{W}$ is an M-bi-base of $\mathbb{S}$, then for any $s \in \mathbb{S}$, there exists $z_1 \in \mathbb{W}$ such that $s \leq_{mb} z_1$.*

Lemma 3.1. *Let $\mathbb{W}$ be an M-bi-base of $\mathbb{S}$. If $z_1, z_2 \in \mathbb{W}$ such that $z_1 \neq z_2$, then neither $z_1 \leq_{mb} z_2$, nor $z_2 \leq_{mb} z_1$.*

Proof: Assume that $z_1, z_2 \in \mathbb{W}$ such that $z_1 \neq z_2$. Suppose that $z_1 \leq_{mb} z_2$. Let $\mathbb{V} = \mathbb{W}\backslash\{z_1\}$. Then $z_2 \in \mathbb{V}$. Let $s \in \mathbb{S}$. By Remark 3.3, there exists $z_3 \in \mathbb{W}$ such that $s \leq_{mb} z_3$. We think about two cases. If $z_3 \neq z_1$, then $z_3 \in \mathbb{V}$; thus, $\langle s\rangle_{mb} \subseteq \langle z_3\rangle_{mb} \subseteq \langle\mathbb{V}\rangle_{mb}$. Hence, $\mathbb{S} = \langle\mathbb{V}\rangle_{mb}$, which is a contradiction. If $z_3 = z_1$, then $s \leq_{mb} z_2$. Hence, $s \in \langle\mathbb{V}\rangle_{mb}$, since $z_2 \in \mathbb{V}$. Hence, $\mathbb{S} = \langle\mathbb{V}\rangle_{mb}$, which is a contradiction. Similarly to prove other case.

Lemma 3.2. *Let $\mathbb{W}$ be an M-bi-base of $\mathbb{S}$ and $z_1, z_2, z_3 \in \mathbb{W}$ and $s \in \mathbb{S}$.*

(1) If $z_1 \in (\{z_2 \times \xi \times z_3\} \cup N(\{z_2 \times \xi \times z_3\} \times \Gamma \times \{z_2 \times \xi \times z_3\}) \cup \{z_2 \times \xi \times z_3\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{z_2 \times \xi \times z_3\}]$, then $z_1 = z_2$ or $z_1 = z_3$,

(2) If $z_1 \in (\{z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3\} \cup N(\{z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3\} \times \Gamma \times \{z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3\}) \cup \{z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3\}]$, then $z_1 = z_2$ or $z_1 = z_3$.

Proof: (1) Suppose that $z_1 \neq z_2$ and $z_1 \neq z_3$. Let $\mathbb{V} = \mathbb{W}\backslash\{z_1\}$. Clearly, $\mathbb{V} \subset \mathbb{W}$. Since $z_1 \neq z_2$ and $z_1 \neq z_3$ imply $z_2, z_3 \in \mathbb{V}$. To prove that $\langle\mathbb{W}\rangle_{mb} \subseteq \langle\mathbb{V}\rangle_{mb}$, it suffices to determine that $\mathbb{W} \subseteq \langle\mathbb{V}\rangle_{mb}$. Let $v \in \mathbb{W}$, if $v \neq z_1$ that $v \in \mathbb{V}$ and hence $v \in \langle\mathbb{V}\rangle_{mb}$. If $v = z_1$, then $v = z_1 \in (\{z_2 \times \xi \times z_3\} \cup N(\{z_2 \times \xi \times z_3\} \times \Gamma \times \{z_2 \times \xi \times z_3\}) \cup \{z_2 \times \xi \times z_3\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{z_2 \times \xi \times z_3\}] \subseteq (\mathbb{V} \times \Gamma \times \mathbb{V}) \cup \mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V} \subseteq \langle\mathbb{V}\rangle_{mb}$. Thus, $\mathbb{W} \subseteq \langle\mathbb{V}\rangle_{mb}$. This implies $\langle\mathbb{W}\rangle_{mb} \subseteq \langle\mathbb{V}\rangle_{mb}$. Since $\mathbb{W}$ is an M-bi-base of $\mathbb{S}$ and $\mathbb{S} = \langle\mathbb{W}\rangle_{mb} \subseteq \langle\mathbb{V}\rangle_{mb} \subseteq \mathbb{S}$. Therefore, $\mathbb{S} = \langle\mathbb{V}\rangle_{mb}$, which is a contradiction. Hence, $z_1 = z_2$ or $z_1 = z_3$.

(2) Suppose that $z_1 \neq z_2$ and $z_1 \neq z_3$. Let $\mathbb{V} = \mathbb{W}\backslash\{z_1\}$. Clearly, $\mathbb{V} \subset \mathbb{W}$. Since $z_1 \neq z_2$ and $z_1 \neq z_3$, imply $z_2, z_3 \in \mathbb{V}$. To prove that $\langle\mathbb{W}\rangle_{mb} \subseteq \langle\mathbb{V}\rangle_{mb}$, it remains to prove that $\mathbb{W} \subseteq \langle\mathbb{V}\rangle_{mb}$. Let $v \in \mathbb{W}$, if $v \neq z_1$ that $v \in \mathbb{V}$ and so $v \in \langle\mathbb{V}\rangle_{mb}$. If $v = z_1$, then $v = z_1 \in (\{z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3\} \cup N(\{z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3\} \times \Gamma \times \{z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3\}) \cup \{z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3\}] \subseteq (\mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}] \subseteq \langle\mathbb{V}\rangle_{mb}$. Thus, $\mathbb{W} \subseteq \langle\mathbb{V}\rangle_{mb}$. This implies $\langle\mathbb{W}\rangle_{mb} \subseteq \langle\mathbb{V}\rangle_{mb}$. Since $\mathbb{W}$ is an M-bi-base of $\mathbb{S}$ and $\mathbb{S} = \langle\mathbb{W}\rangle_{mb} \subseteq \langle\mathbb{V}\rangle_{mb} \subseteq \mathbb{S}$, $\mathbb{S} = \langle\mathbb{V}\rangle_{mb}$, which is a contradiction. Hence, $z_1 = z_2$ or $z_1 = z_3$.

Lemma 3.3. *Let $\mathbb{W}$ be an M-bi-base of $\mathbb{S}$,*

(1) If $z_1 \neq z_2$ and $z_1 \neq z_3$, then $z_1 \nleq_{mb} z_2 \times \xi \times z_3$,

(2) If $z_1 \neq z_2$ and $z_1 \neq z_3$, then $z_1 \nleq_{mb} z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3$, for $z_1, z_2, z_3 \in \mathbb{W}$, $\xi, \xi_i, \nu \in \Gamma$ and $s_i \in \mathbb{S}$, $i = 1, 2, \ldots, n$.

Proof: (1) For any $z_1, z_2, z_3 \in \mathbb{W}$, let $z_1 \neq z_2$ and $z_1 \neq z_3$. Suppose that $z_1 \leq_{mb} z_2 \times \xi \times z_3$ and $z_1 \in \langle z_1\rangle_{mb} \subseteq \{(z_2 \times \xi \times z_3)\}_{mb} = (\{(z_2 \times \xi \times z_3)\} \cup N(\{(z_2 \times \xi \times z_3)\} \times \Gamma \times \{(z_2 \times \xi \times z_3)\}) \cup \{(z_2 \times \xi \times z_3)\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{(z_2 \times \xi \times z_3)\}]$. By Lemma 3.2(1), it follows that $z_1 = z_2$ or $z_1 = z_3$, which is a contradiction.

(2) For any $z_1, z_2, z_3 \in \mathbb{W}$, let $z_1 \neq z_2$ and $z_1 \neq z_3$. Suppose that $z_1 \leq_{mb} z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3$, we have $z_1 \in \langle z_1\rangle_{mb} \subseteq \{(z_2 \times \Gamma \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3)\}_{mb} = (\{(z_2 \times \Gamma \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3)\} \cup N(\{(z_2 \times \Gamma \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3)\} \times \Gamma \times \{(z_2 \times \Gamma \times (s_1 \times \xi_1 \times s_2 \times \cdots \times$

$\xi_n \times s_n) \times \nu \times z_3)\}) \cup \{(z_2 \times \Gamma \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3)\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{(z_2 \times \Gamma \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3)\}]$. By Lemma 3.2(2), it follows that $z_1 = z_2$ or $z_1 = z_3$, which is a contradict to our handling.

Theorem 3.4. *Let $\mathbb{W}$ be an M-bi-base of $\mathbb{S}$ if and only if $\mathbb{W}$ satisfies the following holds.*

(1) For any $s \in \mathbb{S}$,
(1.1) there exists $z_2 \in \mathbb{W}$ such that $s \leq_{mb} z_2$ (or),
(1.2) there exists $w_1, w_2 \in \mathbb{W}$ such that $s \leq_{mb} w_1 \times \xi \times w_2$ (or),
(1.3) there exists $w_3, w_4 \in \mathbb{W}$ such that $s \leq_{mb} w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4$,
(2) If $z_1 \neq z_2$ and $z_1 \neq z_3$, then $z_1 \not\leq_{mb} z_2 \times \xi \times z_3$, for any $z_1, z_2, z_3 \in \mathbb{W}$,
(3) If $z_1 \neq z_2$ and $z_1 \neq z_3$, then $z_1 \not\leq_{mb} z_2 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times z_3$, for any $z_1, z_2, z_3 \in \mathbb{W}$, $s_i \in \mathbb{S}$ and $\xi_i, \xi, \nu \in \Gamma$, $i = 1, 2, \ldots, n$.

Proof: Assuming that $\mathbb{W}$ is an M-bi-base of $\mathbb{S}$, then $\mathbb{S} = \langle \mathbb{W} \rangle_{mb}$. To prove (1), let $s \in \mathbb{S}$, $s \in (\mathbb{W} \cup \mathbb{W} \times \Gamma \times \mathbb{W} \cup \mathbb{W} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{W}]$. We have $s \leq w$ for some $w \in \mathbb{W} \cup \mathbb{W} \times \Gamma \times \mathbb{W} \cup \mathbb{W} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{W}$, and we think about the following three cases.

Case-1: Let $w \in \mathbb{W}$. Then $w = z_2$ for some $z_2 \in \mathbb{W}$. This implies $\langle w \rangle_{mb} \subseteq \langle z_2 \rangle_{mb}$. Hence, $w \leq_{mb} z_2$. Since $s \leq w$ for some $w \in \langle z_2 \rangle_{mb}$, to find out $\langle s \rangle_{mb} \subseteq \langle z_2 \rangle_{mb}$, now, $s \cup N(s \times \Gamma \times s) \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times s \subseteq \langle z_2 \rangle_{mb} \cup N(\langle z_2 \rangle_{mb} \times \Gamma \times \langle z_2 \rangle_{mb}) \cup \langle z_2 \rangle_{mb} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \langle z_2 \rangle_{mb} \subseteq z_2 \cup N(z_2 \times \xi \times z_2) \cup z_2 \times \xi \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2$. We have $(s \cup N(s \times \Gamma \times s) \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times s] \subseteq (z_2 \cup N(z_2 \times \xi \times z_2) \cup z_2 \times \xi \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2]$. Thus, $\langle s \rangle_{mb} \subseteq \langle z_2 \rangle_{mb}$ and hence $s \leq_{mb} z_2$.

Case-2: Let $w \in \mathbb{W} \times \Gamma \times \mathbb{W}$. Then $w = w_1 \times \xi \times w_2$ for some $w_1, w_2 \in \mathbb{W}$ and $\xi \in \Gamma$. This implies $\langle w \rangle_{mb} \subseteq \langle w_1 \times \xi \times w_2 \rangle_{mb}$. Hence, $w \leq_{mb} w_1 \times \xi \times w_2$. Since $s \leq w$ for some $w \in \langle w_1 \times \xi \times w_2 \rangle_{mb}$, we have $s \in \langle w_1 \times \xi \times w_2 \rangle_{mb}$. We determine that $\langle s \rangle_{mb} \subseteq \langle w_1 \times \xi \times w_2 \rangle_{mb}$. Now, $s \cup N(s \times \Gamma \times s) \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times s \subseteq \langle w_1 \times \xi \times w_2 \rangle_{mb} \cup N(\langle w_1 \times \xi \times w_2 \rangle_{mb} \times \Gamma \times \langle w_1 \times \xi \times w_2 \rangle_{mb}) \cup \langle w_1 \times \xi \times w_2 \rangle_{mb} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \langle w_1 \times \xi \times w_2 \rangle_{mb} \subseteq (\{w_1 \times \xi \times w_2\} \cup N(\{w_1 \times \xi \times w_2\} \times \Gamma \times \{w_1 \times \xi \times w_2\}) \cup \{w_1 \times \xi \times w_2\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{w_1 \times \xi \times w_2\}]$. Hence, $(s \cup N(s \times \Gamma \times s) \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times s] \subseteq (\{w_1 \times \xi \times w_2\} \cup N(\{w_1 \times \xi \times w_2\} \times \Gamma \times \{w_1 \times \xi \times w_2\}) \cup \{w_1 \times \xi \times w_2\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{w_1 \times \xi \times w_2\}]$. This implies $\langle s \rangle_{mb} \subseteq \langle w_1 \times \xi \times w_2 \rangle_{mb}$. Hence, $s \leq_{mb} w_1 \times \xi \times w_2$.

Case-3: Let $w \in \mathbb{W} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{W}$. Then $w = w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4$ for some $w_3, w_4 \in \mathbb{W}$. This implies $\langle w \rangle_{mb} \subseteq \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}$. Hence, $w \leq_{mb} \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}$. Since $s \leq w$ for some $w \in \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}$, we have $s \in \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}$. To prove that $\langle s \rangle_{mb} \subseteq \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}$. Now, $s \cup N(s \times \Gamma \times s) \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times s \subseteq \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb} \cup N(\langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb} \times \Gamma \times \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}) \cup \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb} \subseteq (\{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \cup N(\{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \times \Gamma \times \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\}) \cup \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\}]$. Hence, $(s \cup N(s \times \Gamma \times s) \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times s] \subseteq (\{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \cup N(\{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \times \Gamma \times \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\}) \cup \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\}]$. This implies $\langle s \rangle_{mb} \subseteq \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}$. Hence,

$s \leq_{mb} w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4$. By Lemma 3.3(1) and Lemma 3.3(2), prove (2) and (3), respectively.

Conversely, assume that (1), (2) and (3) hold. To prove that $\mathbb{W}$ is an M-bi-base of $\mathbb{S}$. Determine that $\mathbb{S} = \langle \mathbb{W} \rangle_{mb}$. Clearly, $\langle \mathbb{W} \rangle_{mb} \subseteq \mathbb{S}$. By (1), $\mathbb{S} \subseteq \langle \mathbb{W} \rangle_{mb}$ and $\mathbb{S} = \langle \mathbb{W} \rangle_{mb}$. It remains to find out $\mathbb{W}$ is a minimal subset of $\mathbb{S}$, $\mathbb{S} = \langle \mathbb{W} \rangle_{mb}$. Suppose that $\mathbb{S} = \langle \mathbb{V} \rangle_{mb}$ for some $\mathbb{V} \subset \mathbb{W}$. Since, $\mathbb{V} \subset \mathbb{W}$, there exists $z_2 \in \mathbb{W} \setminus \mathbb{V}$. Since $z_2 \in \mathbb{W} \subseteq \mathbb{S} = \langle \mathbb{V} \rangle_{mb}$ and $z_2 \notin \mathbb{V}$, it follows that $z_2 \in (\mathbb{V} \times \Gamma \times \mathbb{V} \cup \mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}]$. Since $z_2 \in (\mathbb{V} \times \Gamma \times \mathbb{V} \cup \mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}]$, it implies $z_2 \leq w$ for some $w \in \mathbb{V} \times \Gamma \times \mathbb{V} \cup \mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}$. There are two cases to observe.
Case-1: Let $w \in \mathbb{V} \times \Gamma \times \mathbb{V}$. Then $w = w_1 \times \xi \times w_2$ for some $w_1, w_2 \in \mathbb{V}$ and $\xi \in \Gamma$. We have $w_1, w_2 \in \mathbb{W}$. Since $z_2 \notin \mathbb{V}$, $z_2 \neq w_1$ and $z_2 \neq w_2$. Since $w = w_1 \times \xi \times w_2$, $\langle w \rangle_{mb} \subseteq \langle w_1 \times \xi \times w_2 \rangle_{mb}$. Hence, $w \leq_{mb} w_1 \times \xi \times w_2$. Since $z_2 \leq w$ for some $w \in \langle w_1 \times \xi \times w_2 \rangle_{mb}$, we have $z_2 \in \langle w_1 \times \xi \times w_2 \rangle_{mb}$. To prove that $\langle z_2 \rangle_{mb} \subseteq \langle w_1 \times \xi \times w_2 \rangle_{mb}$. Now, $z_2 \cup N(z_2 \times \xi \times z_2) \cup z_2 \times \xi \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2 \subseteq \langle w_1 \times \xi \times w_2 \rangle_{mb} \cup N(\langle w_1 \times \xi \times w_2 \rangle_{mb} \times \Gamma \times \langle w_1 \times \xi \times w_2 \rangle_{mb}) \cup \langle w_1 \times \xi \times w_2 \rangle_{mb} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \langle w_1 \times \xi \times w_2 \rangle_{mb} \subseteq (\{w_1 \times \xi \times w_2\} \cup N(\{w_1 \times \xi \times w_2\} \times \Gamma \times \{w_1 \times \xi \times w_2\}) \cup \{w_1 \times \xi \times w_2\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{w_1 \times \xi \times w_2\}]$. Hence, $(z_2 \cup N(z_2 \times \xi \times z_2) \cup z_2 \times \xi \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2] \subseteq (\{w_1 \times \xi \times w_2\} \cup N(\{w_1 \times \xi \times w_2\} \times \Gamma \times \{w_1 \times \xi \times w_2\}) \cup \{w_1 \times \xi \times w_2\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{w_1 \times \xi \times w_2\}]$. This implies $\langle z_2 \rangle_{mb} \subseteq \langle w_1 \times \xi \times w_2 \rangle_{mb}$. Hence, $z_2 \leq_{mb} w_1 \times \xi \times w_2$. This contradicts (2).
Case-2: Let $w \in \mathbb{V} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \mathbb{V}$. Then $w = w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4$ for some $w_3, w_4 \in \mathbb{V}$, $s_i \in \mathbb{S}$ and $\xi_i, \xi, \nu \in \Gamma$, $i = 1, 2, \ldots, n$. We have $w_3, w_4 \in \mathbb{W}$. Since $z_2 \notin \mathbb{V}$, $z_2 \neq w_3$ and $z_2 \neq w_4$. Since $w = w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4$, $\langle w \rangle_{mb} \subseteq \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}$. Hence, $w \leq_{mb} w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4$. Since $z_2 \leq w$ for some $w \in \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}$, we have $z_2 \in \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}$. We determine that $\langle z_2 \rangle_{mb} \subseteq \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}$. Now, $z_2 \cup N(z_2 \times \xi \times z_2) \cup z_2 \times \xi \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2 \subseteq \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb} \cup N(\langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb} \times \Gamma \times \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}) \cup \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb} \subseteq (\{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \cup N(\{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \times \Gamma \times \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\}) \cup \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \nu \times w_4\}]$. Hence, $(z_2 \cup N(z_2 \times \xi \times z_2) \cup z_2 \times \xi \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times z_2] \subseteq (\{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \cup N(\{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \times \Gamma \times \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\}) \cup \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\} \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times \{w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4\}]$. This implies $\langle z_2 \rangle_{mb} \subseteq \langle w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4 \rangle_{mb}$. Hence, $z_2 \leq_{mb} w_3 \times \xi \times (s_1 \times \xi_1 \times s_2 \times \cdots \times \xi_n \times s_n) \times \nu \times w_4$, which is a contradiction to (3). Therefore, $\mathbb{W}$ is an M-bi-base of $\mathbb{S}$.

Theorem 3.5. *Let $\mathbb{W}$ be an M-bi-base of $\mathbb{S}$. Then $\mathbb{W}$ is an ordered Γ-subsemigroup of $\mathbb{S}$ if and only if $w_1 \times \xi \times w_2 = w_1$ or $w_1 \times \xi \times w_2 = w_2$, for any $w_1, w_2 \in \mathbb{W}$ and $\xi \in \Gamma$.*

Proof: If $\mathbb{W}$ is an ordered Γ-subsemigroup of $\mathbb{S}$, then $w_1 \times \xi \times w_2 \in \mathbb{W}$. Since $w_1 \times \xi \times w_2 \in (N(w_1 \times \Gamma \times w_2) \cup w_1 \times \Gamma \times (\mathbb{S} \times \Gamma \times \cdots \times \Gamma \times \mathbb{S}) \times \Gamma \times w_2]$, it follows by Lemma 3.3 that $w_1 \times \xi \times w_2 = w_1$ or $w_1 \times \xi \times w_2 = w_2$.

4. **Conclusion.** We have introduced the M-bi-base of an ordered Γ-semigroup and discussed some characterizations of the M-bi-base. We have discussed some of their basic properties and characterized some of their properties using M-bi-ideal and its generator.

It was also presented the M-base of an ordered Γ-semigroup generated by an element and a subset. In the future, we will characterize some more classes of the Γ-semigroup and ordered Γ-semigroup based on M-left-base, and M-right-base, respectively. Moreover, some other classes of the various tri-bases and various tri-M-bases will be studied. Their study with regard to the ordered Γ-hyper semigroup based on bi-base and M-bi-base will be explored.

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