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NON-POLYNOMIAL SPLINE FUNCTIONS FOR SOLVING TWO-DIMENSIONAL LINEAR FUZZY FREDHOLM INTEGRAL EQUATIONS

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Abstract. *The aim of this paper is solving two-dimensional fuzzy Fredholm integral equation of the second kind (2D-FFIE2), by proposing cubic non-polynomial spline functions. The fuzzy equation is converted into two crisp equations, and then the dual system is solved by the proposed spline. Convergence is proved by iteration method. It is accurate and easy to apply with Math Cad 15 computation for numerical examples.*
Keywords: Non-polynomial spline, Two-dimensional fuzzy Fredholm integral equation, Picard's iterations, Convergence analysis

1. **Introduction.** Studying fuzzy integral equations is important in solving a large proportion of the problems in many topics in applied mathematics, particularly in relation to physics, geography, medicine, and biology. Usually in many applications, some of the parameters in our problems are represented by fuzzy number rather than crisp, and hence it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy integral equations and solve them (see for example, [1-3]). In connection with the application, it is appropriate to study the existence and uniqueness of solutions by fixed point theorems and iteration methods as in [4-6].

Numerical methods were proposed to solve 2D-FFIE2. For instance, in [7] Ezzati and Ziari determined bivariate Bernstein polynomials. In [8] Farshid et al. used triangular functions. In [9,10] Nouriani and Ezzati introduced interpolation methods, fuzzy Lagrange and fuzzy bicubic spline respectively to find numerical results. In [11] Ameri and Nezhad solved fuzzy Volterra integral equations by least square approximation method.

For non-polynomial spline, in [12] Hasan presented cubic non-polynomial spline to solve nonlinear Volterra integrodifferential equations. In [13] Ding and Wang applied mid-knot cubic spline to obtaining numerical solution of second-order boundary value problems. In [14,15] Hasan and Salim proposed linear non-polynomial spline to get accurate results for solving fractional partial differential equations and two-dimensional variable order fractional derivative, respectively.

This paper develops a simple but quite accurate numerical method for approximating the solution of 2D-FFIE2 by non-polynomial spline functions.

The outline of this paper is organized as follows. Some basic definitions of fuzzy set theory and non-polynomial spine are reviewed in Section 2. In Section 3, the method is constructed. In Section 4, convergence analysis is verified. For application, two numerical examples are presented in Section 5. Conclusions are drawn in Section 6.

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2. Preliminaries.

2.1. Basic concepts of fuzzy set theory.
Basic definitions, properties and mathematical operations of fuzzy number valued functions are reviewed.

Definition 2.1. [16] *A fuzzy number is a function $u\colon \mathbb{R} \to [0,1]$ satisfying the following:*
1) $u(x) = 0$ outside of some interval $[0,1] \subset \mathbb{R}$;
2) u is a fuzzy convex set, that is for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$, $u(\lambda x + (1-\lambda)y) \geq \min(u(x), u(y))$.

The set of all fuzzy numbers is denoted by $\mathbb{R}_f$, the fuzzy number $r \in [0,1]$ in parametric form is denoted by ordered pair of functions $(\underline{u}(r), \overline{u}(r))$ such that
1) $\underline{u}(r)$ nondecreasing function bounded on $[0,1]$;
2) $\overline{u}(r)$ nonincreasing function bounded on $[0,1]$;
3) $\underline{u}(r) \leq \overline{u}(r)$.

The addition and multiplication [17] operations of real numbers can be extended to fuzzy numbers. For all $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k \in \mathbb{R}$, we have
1) $u = v$ iff $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$;
2) $u \oplus v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$;
3) $k \odot u = \begin{cases} k\underline{u}(r), k\overline{u}(r), & k \geq 0 \\ k\overline{u}(r), k\underline{u}(r), & k < 0 \end{cases}$.

As a distance between fuzzy numbers, we use Hausdorff metric defined by [18]
$$D(u,v) = \sup_{r \in [0,1]} \max \{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) + \overline{v}(r)|\}$$

Lemma 2.1. [18] *The Hausdorff metric has the following properties*
1) $(\mathbb{R}_f, D)$ is complete matric space;
2) $D(u \oplus w, v \oplus w) = D(u,v)$ for all $u, v, w \in \mathbb{R}_f$;
3) $D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e)$ for all $u, v, w, e \in \mathbb{R}_f$;
4) $D(k \odot u, k \odot v) = |k| D(u,v)$ for all $u, v \in \mathbb{R}_f$ and $k \in \mathbb{R}$.

Definition 2.2. [19] *Distance between fuzzy functions f, g such that $f, g\colon [a,b] \to \mathbb{R}_f$ is fuzzy real-valued functions defined by D_u such that*
$$D_u = \sup\{D(f(x), g(x)) | x \in [a,b]\}$$

Definition 2.3. [19] *If for every $\epsilon > 0$, $\exists \delta > 0$ such that $D(f(x), g(x)) < \epsilon$ whenever $x \in [a,b]$, $|x - x_0| < \delta$ then $f\colon [a,b] \to \mathbb{R}_f$ is referred to as fuzzy continuous function at $x_0 \in [a,b]$.*

2.2. A review of one-dimensional non-polynomial spline.
The partition $\Delta = \{x_0, x_1, \dots, x_n\}$ of $[a,b] \in \mathbb{R}$. Let S_Δ denote the set of piecewise polynomials on subinterval $I_i = [x_i, x_{i+1}]$ of partition Δ. Consider the grid point x_i on interval $[a,b]$ such that
$$a = x_0 < x_1 < \cdots < x_n = b \text{ with } x_i = x_0 + ih, \quad i = 0, 1, \dots, n \text{ and } h = \frac{b-a}{n}$$

The non-polynomial spline function obtained by the segment $S_i(x)$ has the form: $\cos kx + \sin kx + P_{n-2}(x)$, where $P_{n-2}(x) = \sum_{i=0}^{n-2} l_i x^i$ polynomial of degree $n-2$ and a trigonometric part $\cos kx$ and $\sin kx$ with a free parameter k.

Definition 2.4. [20] *For each segment $[x_i, x_{i+1}]$, $i = 0, 1, \dots, n-1$, the cubic non-polynomial $S_i(x)$ has the form $a_i \cos k(x - x_i) + b_i \sin k(x - x_i) + c_i(x - x_i) + d_i$ where a_i, b_i, c_i, d_i are constant coefficients.*

3. **The Method.** We define two dimensional nonpolynomial spline functions as follows:

$$S_{m,n}(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}c_{ij}S_i(x)\otimes S_j(y),\quad \forall(x,y)\in[a,b]^2,\ \forall(m,n)\in\mathbb{N}^2 \tag{1}$$

where $\otimes$ denotes the Kronecker product of $S_i(x)$, $S_j(y)$ and c_{ij} are constant coefficients. $S_i(x)$ and $S_j(y)$ are cubic non-polynomial spline as in Definition 2.4.

The 2D-FFIE2 is defined by

$$u(x,y)=f(x,y)\oplus\lambda\otimes\int_a^b\int_a^b K(x,y,s,t)\otimes u(s,t)dsdt,\quad (x,y)\in E \tag{2}$$

where $\lambda>0$, K is crisp given function on $E\times E$, where $E=[a,b]^2$. $u(x,y)$ and $f(x,y)$ are continuous fuzzy real-valued functions.

By the parametric form, let $\left(\underline{f}(x,y,r),\overline{f}(x,y,r)\right)$ and $(\underline{u}(x,y,r),\overline{u}(x,y,r))$, $0\le r\le 1$, be parametric form of fuzzy functions $f(x,y)$ and $u(x,y)$, respectively. Substituting these forms into (2), we have

$$\begin{aligned}&(\underline{u}(x,y,r),\overline{u}(x,y,r))\\ &=\left(\underline{f}(x,y,r),\overline{f}(x,y,r)\right)\oplus\lambda\otimes\int_a^b\int_a^b K(x,y,s,t)\otimes(\underline{u}(s,t,r),\overline{u}(s,t,r))\,dsdt\end{aligned} \tag{3}$$

By substituting (1) in (2) with $a=0$, $b=1$, $\lambda=1$ and $m=n$, we have

$$\begin{aligned}&\sum_{i=0}^{n}\sum_{j=0}^{n}c_{ij}S_i(x)\otimes S_j(y)\\ &=f(x,y)\oplus\int_0^1\int_0^1 K(x,y,s,t)\otimes\sum_{i=0}^{n}\sum_{j=0}^{n}c_{ij}S_i(s)\otimes S_j(t)dsdt\end{aligned}$$

By letting $\sum_{i=0}^{n}\sum_{j=0}^{n}c_{ij}S_i(x)\otimes S_j(y)=S_{ij}(x,y)$, then the approximate solution is

$$S_{ij}(x,y)=f(x,y)\oplus\int_0^1\int_0^1 K(x,y,s,t)\otimes S_{ij}(s,t)dsdt \tag{4}$$

The approximate solution $S_{ij}(x,y)$ at arbitrary points of Equation (4) can be written in the form

$$\begin{aligned}\overline{S}_{ij}(x,y)&=\overline{f}(x,y)\oplus\int_0^1\int_0^1 K(x,y,s,t)\otimes\overline{S}_{ij}(s,t)dsdt\\ \underline{S}_{ij}(x,y)&=\underline{f}(x,y)\ \oplus\int_0^1\int_0^1 K(x,y,s,t)\otimes\underline{S}_{ij}(s,t)dsdt\end{aligned}$$

If we substitute x and y with assumed collocation points $x_p=\frac{p+0.2}{n+1}$, $p=0,1,2,\ldots,n$ and $y_q=\frac{q+0.2}{q}$, $q=0,1,2,\ldots,n$, respectively.

Then (4) can be represented by dual fuzzy linear system $AC=F$. $A=a_{ij}^{pq}$, $p,q=0,1,2,\ldots,n$, and $i,j=0,1,2,\ldots,n$, is $(n+1)^2\times(n+1)^2$ fuzzy matrix, where $a_{ij}^{pq}=S_i(x_p)\otimes S_j(y_q)$. F is $(n+1)^2\times 1$ fuzzy vectors, since $F=(f(0,0)\ldots f(0,1)\ldots f(1,0)\ldots f(1,1))^T$. $C=c_{ij}$, $i,j=0,1,\ldots,n$ are unknown coefficients to be determined.

By solving the dual fuzzy linear system (4), the unknown coefficients c_{ij} can be found. So, we obtain the approximate solution of 2D-FFIE2.

4. **Convergence Theorem.** Convergence for the numerical solution of two-dimensional Volterra integral equations is verified by Katani and Shahmorad in [21] by using Gronwall inequality. In this section, the convergence and uniqueness of fuzzy solution for 2D-FFIE2 are proved by using Picard's iterative method. In the following theorem, for a given value $u_0(x,y,r)$, the Picard's iteration for (3) is defined by

$$u_{n+1}(x,y,r) = f(x,y,r) + \lambda \int_c^d \int_a^b K(x,y,s,t) u_n(s,t,r) ds dt, \quad n = 0,1,\ldots \tag{5}$$

where $u(x,y,r) = (\underline{u}(x,y,r), \overline{u}(x,y,r))$ and $f(x,y,r) = \big(\underline{f}(x,y,r), \overline{f}(x,y,r)\big)$

Theorem 4.1. *Let $f(x,y,r)$ be a fuzzy continuous and bounded over E, and let $K(x,y,s,t)$ be a continuous function and satisfies Lipschitz condition L over $E \times E$. Then a unique fuzzy solution of (3) and the Picard's iteration are uniformly convergent to exact solution $u(x,y,r)$.*

Proof: Since $K(x,y,s,t)$ is continuous over $E \times E$, there exists a constant $M > 0$, such that $|K(x,y,s,t)u(s,t,r)| \leq M$, if $u(x,y,r)$ is bounded say, $\alpha < u(x,y,r) < \beta$. To ensure that

$$|u_{n+1}(x,y,r) - f(x,y,r)| \leq |\lambda| \int_c^d \int_a^b |K(x,y,s,t) u_n(s,t,r)| ds dt \leq |\lambda|\, MST$$

where $S = d - c$ and $T = b - a$, then $u_{n+1}(x,y,r)$ is bounded. By the bounds of $f(x,y,r)$, there exist integers m_1 and m_2, such that $m_1 < f(x,y,r) < m_2$; thus, we have

$$\alpha < m_1 - |\lambda| MST < |u_{n+1}(x,y,r)| < |\lambda| MST + m_2 < \beta$$

Now, we follow a procedure to get convergence of the sequence $u_n(s,t,r)$ to the unique solution of (3). by the Picard's iteration in (5), we have for $n = 1$

$$\begin{aligned}
|u_2(x,y,r) - u_1(x,y,r)| &= |\lambda| \int_c^d \int_a^b |K(x,y,s,t)u_1(s,t,r) - K(x,y,s,t)u_0(s,t,r)| \\
&\leq |\lambda| L \int_c^d \int_a^b |u_1(s,t,r) - u_0(s,t,r)| ds dt \\
&\leq |\lambda| L |\beta - \alpha| ST
\end{aligned}$$

for $n = m$

$$|u_{m+1}(x,y,r) - u_m(x,y,r)| \leq |\beta - \alpha| \frac{(|\lambda| LST)^m}{m! m!}$$

for $n = m + 1$

$$\begin{aligned}
|u_{m+2}(x,y,r) - u_{m+1}(x,y,r)| &\leq |\lambda| L \int_c^d \int_a^b |u_{m+1}(s,t,r) - u_m(s,t,r)| ds dt \\
&\leq |\lambda| L \int_c^d \int_a^b |\beta - \alpha| \frac{(|\lambda| LST)^m}{m! m!} ds dt \\
&\leq |\beta - \alpha| \frac{(|\lambda| LST)^{m+1}}{(m+1)!(m+1)!}
\end{aligned} \tag{6}$$

In (6) replace $m + 1$ by n, then

$$|u_{n+1}(x,y,r) - u_n(x,y,r)| \leq |\beta - \alpha| \frac{(|\lambda| LST)^n}{n! n!} \quad \forall\, n \in N$$

which implies that the series: $\sum_{n=1}^{\infty}(u_{n+1}(x,y,r) - u_n(x,y,r))$ is absolutly and uniformly convergent. On the other hand, $u(x,y,r) = u_1(x,y,r) + \sum_{i=1}^{n-1} u_{i+1}(x,y,r) - u_i(x,y,r)$. Then $\lim_{n\to\infty} u_n(x,y,r)$ exists for all $(x,y) \in E$. Let $\lim_{n\to\infty} u_n(x,y,r) = u(x,y,r)$. Then, we have

$$\lim_{n\to\infty} K(x,y,s,t) u_n(x,y,r) = K(x,y,s,t) u(x,y,r)$$

and so

$$\lim_{n\to\infty} u_n(x,y,r) = f(x,y) + \lambda \int_c^d \int_a^b K(x,y,s,t)u(s,t,r)dsdt = u(x,y,r)$$

That is, $u(x,y,r)$ is the unique solution of (3).

5. **Numerical Examples.** To solve 2D-FFIE2 by the proposed method, two examples are tested. Tables 1 and 2 show the exact solution and numerical solution with absolute error $\underline{\xi} = |\underline{u} - \underline{S}|$, $\overline{\xi} = \left|\overline{u} - \overline{S}\right|$ of exact and approximate solutions. In Example 5.1, we compare the results of the proposed method with method of [7] and for Example 5.2 with method of [22].

Example 5.1. *Consider the following 2D-FFIE2 in (3) with*

$$\underline{f}(x,y,r) = x\sin\left(\frac{y}{2}\right)\left(r^2+r\right), \quad \overline{f}(x,y,r) = x\sin\left(\frac{y}{2}\right)\left(4-r^3-r\right)$$

$\lambda = 1$, *and* $K(x,y,s,t) = x^2ys$ *for* $0 \le x,y,s,t \le 1$. *The exact solution [7] is*

$$\underline{u}(x,y,r) = \left(x\sin\left(\frac{y}{2}\right) - \frac{16}{21}\left(\cos\left(\frac{1}{2}\right) - 1\right)x^2y\right)\left(r^2+r\right)$$

$$\overline{u}(x,y,r) = \left(x\sin\left(\frac{y}{2}\right) - \frac{16}{21}\left(\cos\left(\frac{1}{2}\right) - 1\right)x^2y\right)\left(4-r^3-r\right)$$

From Table 1, we can find that the absolute errors $\underline{\xi}$ *and* $\bar{\xi}$ *of the numerical solutions are in good agreement with exact solution. By comparing the proposed method with fuzzy bivariate Bernstein polynomials [7], we see that the proposed method has a higher accuracy and much smaller error with less collocation points.*

TABLE 1. Numerical results for Example 5.1 in $x = 0.3$, $y = 0.6$

	Exact solution	Approximate solution	Errors of the present method		Errors of method [7]	
r	$\underline{u}(x,y,r)$, $\overline{u}(x,y,r)$	$\underline{S}_{ij}(x,y,r)$, $\overline{S}_{ij}(x,y,r)$	$\underline{\xi}$	$\overline{\xi}$	$\underline{\xi}$	$\overline{\xi}$
0.0	$(0.000, 0.375)$	$(0.000, 0.375)$	0	1.874×10^{-4}	0.0000	0.0002
0.2	$(0.022, 0.355)$	$(0.022, 0.355)$	1.124×10^{-5}	1.776×10^{-4}	0.0000	0.0002
0.4	$(0.052, 0.331)$	$(0.052, 0.331)$	2.623×10^{-5}	1.656×10^{-4}	0.0000	0.0002
0.6	$(0.090, 0.298)$	$(0.090, 0.298)$	4.497×10^{-5}	1.492×10^{-4}	0.0001	0.0002
0.8	$(0.135, 0.252)$	$(0.135, 0.252)$	6.746×10^{-5}	1.259×10^{-4}	0.0001	0.0002
1.0	$(0.187, 0.187)$	$(0.187, 0.187)$	4.213×10^{-5}	9.369×10^{-5}	0.0001	0.0001

Example 5.2. *Consider the following 2D-FFIE2 in (3) with*

$$\underline{f}(x,y,r) = (2r\cos(1-r)-1)\left(1+x^2+y-\frac{13}{24}(x+y)\right)$$

$$\overline{f}(x,y,r) = \left(2-\sin\left(\frac{r\pi}{2}\right)\right)\left(1+x^2+y-\frac{13}{24}(x+y)\right)$$

$\lambda = 1$, *and* $K(x,y,s,t) = (x+y)st$ *for* $0 \le x,y,s,t \le 1$. *The exact solution [22] is*

$$\underline{u}(x,y,r) = (2r\cos(1-r)-1)\left(x^2+y+1\right)$$

$$\overline{u}(x,y,r) = \left(2-\sin\left(\frac{r\pi}{2}\right)\right)\left(x^2+y+1\right)$$

From Table 2 we can find that the absolute errors $\underline{\xi}$ and $\bar{\xi}$ of the numerical solutions are in good agreement with exact solution. By comparing the proposed method with fuzzy bivariate Bernstein polynomials [22], we see that the proposed method has a higher accuracy.

TABLE 2. Numerical results for Example 5.2 in $x = 0.5$, $y = 0.5$

	Exact solution	Approximate solution	Errors of the present method		Errors of method [22]	
r	$\underline{u}(x,y,r)$, $\overline{u}(x,y,r)$	$\underline{S}_{ij}(x,y,r)$, $\overline{S}_{ij}(x,y,r)$	$\underline{\xi}$	$\overline{\xi}$	$\underline{\xi}$	$\overline{\xi}$
0.0	$(-1.750, 3.500)$	$(-1.750, 3.499)$	4.547×10^{-4}	9.095×10^{-4}	1.0×10^{-3}	2.2×10^{-3}
0.2	$(-1.262, 2.959)$	$(-1.265, 2.958)$	3.28×10^{-4}	7.689×10^{-4}	7.8×10^{-4}	6.4×10^{-3}
0.4	$(-0.595, 2.471)$	$(-0.594, 2.471)$	1.545×10^{-4}	6.422×10^{-4}	1.3×10^{-4}	8.1×10^{-3}
0.6	$(0.184, 2.084)$	$(0.184, 2.084)$	4.787×10^{-4}	5.416×10^{-4}	6.1×10^{-4}	7.3×10^{-3}
0.8	$(0.994, 1.836)$	$(0.994, 1.835)$	2.583×10^{-4}	4.77×10^{-4}	1.8×10^{-3}	4.7×10^{-3}
1.0	$(1.750, 1.750)$	$(1.750, 1.750)$	4.547×10^{-4}	4.547×10^{-4}	1.0×10^{-3}	4.0×10^{-3}

6. **Conclusion.** In this work, non-polynomial spline method approaches the solution of linear 2D-FFIE2. By this method, the original equation is converted into two crisp 2D-FFIE2. Convergence of the exact solution and uniqueness are proved by Picard's iterations in Theorem 4.1. The efficiency of this method is illustrated by tables of the numerical examples which is compared with results of methods in [7] and [22].

This idea can be continued in studying other types of fuzzy integral equations, such as nonlinear equations, or equation with singular kernel.

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