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DIFFERENT APPROACH TOWARDS ORDERED ALMOST IDEALS OF ORDERED SEMIRINGS

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Abstract. *Various ordered almost ideals (ordered $\mathcal{A}$-ideals), including ordered quasi $\mathcal{A}$-ideals, ordered bi quasi $\mathcal{A}$-ideals, ordered tri $\mathcal{A}$-ideals, and ordered tri quasi $\mathcal{A}$-ideals in ordered semirings, are introduced in the current communication along with certain characterizations. We create the implications such as ordered ideals $\Longrightarrow$ ordered quasi ideals $\Longrightarrow$ ordered bi quasi ideals $\Longrightarrow$ ordered tri quasi ideals $\Longrightarrow$ ordered tri quasi $\mathcal{A}$-ideals $\Longrightarrow$ ordered bi quasi $\mathcal{A}$-ideals $\Longrightarrow$ ordered bi $\mathcal{A}$-ideals $\Longrightarrow$ ordered quasi $\mathcal{A}$-ideals $\Longrightarrow$ ordered $\mathcal{A}$-ideals and reverse implications do not hold with examples. We show that the union of ordered $\mathcal{A}$-ideals (bi $\mathcal{A}$-ideals, quasi $\mathcal{A}$-ideals, bi quasi $\mathcal{A}$-ideals) is an ordered $\mathcal{A}$-ideal (bi $\mathcal{A}$-ideal, quasi $\mathcal{A}$-ideal, bi quasi $\mathcal{A}$-ideal) in ordered semirings.*
Keywords: Ordered $\mathcal{A}$-ideals, Ordered bi $\mathcal{A}$-ideals, Ordered quasi $\mathcal{A}$-ideals, Ordered tri $\mathcal{A}$-ideals

1. **Introduction.** The concept of semirings was first proposed as a generalization of rings [1]. The concept of a quasi-ideal in semigroups and rings was introduced by Steinfeld [2]. The semirings were described by Shabir et al. using the characteristics of quasi-ideals [3]. Several authors have described the quasi-ideals of various kinds of semirings [4, 5]. A concept of bi-ideals in semigroups was first proposed by Lajos [6]. The concept of ordered bi-ideals is a generalization of ordered left ideals and ordered right ideals [7, 8]. Many mathematicians used various ideals to demonstrate significant findings and algebraic structural characterizations. The concept of tri-ideals is a generalization of quasi-ideals, bi-ideals, and ideals of semirings [9]. In 2011, Gan and Jiang [10] presented the idea of an

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ordered semiring, which they defined as a semiring with a partially ordered relation on semirings that is compatible with the semiring operations. An ordered semiring S is to be regular in 2014 by Mandal [11] if for any $a \in S$, there exists $x \in S$ such that $a \leq axa$. A relation "$\leq$" that satisfies the requirements of reflexivity, antisymmetry, and transitivity is referred to as a partial order. An ordered semiring is created on a semiring by building a generalized ordinary semiring with a partially ordered relation that is compatible with the operations. Many mathematicians used various ideals to demonstrate significant findings and algebraic structural characterizations. The concept of the semigroup $\mathcal{A}$-ideal was first developed by Grosek and Satko [12]. In 2021, Palanikumar and Arulmozhi [13, 14, 15] discussed several ideals based on semirings. Recently, Palanikumar and Arulmozhi [16, 17] interacted various ideals using semigroups. In 2022, Palanikumar et al. discussed many applications [18, 19]. We provide certain properties of various ordered $\mathcal{A}$-ideals in ordered semirings in this work, to explore a number of significant $\mathcal{A}$-ideal results for ordered semirings and characterize them using quasi $\mathcal{A}$-ideals and bi $\mathcal{A}$-ideals. The following five sections make up the organization of the paper. An introduction is presented in Section 1. An overview of the ordered semirings data is provided in Section 2. The ordered $\mathcal{A}$-ideal was presented in Section 3 with examples. Section 4 provides an ordered tri $\mathcal{A}$-ideal with examples. The conclusion for various $\mathcal{A}$-ideals is provided in Section 5. In this article, we have three goals in mind.

1) To investigate the connection between ordered bi $\mathcal{A}$-ideals and ordered quasi $\mathcal{A}$-ideals.
2) To describe ordered tri $\mathcal{A}$-ideals.
3) To describe ordered bi quasi ideals and ordered tri quasi ideals.

2. **Background.**

Definition 2.1. [7, 8] *A semiring is an algebraic structure $(\mathcal{S}, +, \cdot)$ such that $(\mathcal{S}, +)$ and $(\mathcal{S}, \cdot)$ are semigroups which are connected by the distributive laws. An ordered semiring is a system $(\mathcal{S}, +, \cdot, \leq)$ such that $(\mathcal{S}, +, \cdot)$ is a semiring, $(\mathcal{S}, \leq)$ is a partially ordered set, and for any $a, b, x \in \mathcal{S}$ the following conditions are satisfied.*
(i) If $a \leq b$, then $a + x \leq b + x$ and $x + a \leq x + b$.
(ii) If $a \leq b$, then $ax \leq bx$ and $xa \leq xb$.
An ordered semiring $\mathcal{S}$ is called additively commutative if $a + b = b + a$, for all $a, b \in \mathcal{S}$.

Definition 2.2. [8] *Suppose that I is a nonempty subset of an ordered semiring $(\mathcal{S}, +, \cdot, \leq)$. Then I is called an ordered right (left) ideal of $\mathcal{S}$, if $(I, +)$ is a subsemigroup of $(\mathcal{S}, +)$,*
(a) I is a right (left) ideal of $\mathcal{S}$.
(b) If $x \leq i$ for some $i \in I$, then $x \in I$ (i.e., $I = (I]$).
I is called an ordered ideal if I is an ordered right ideal and ordered left ideal of $\mathcal{S}$.

Definition 2.3. [8] *Suppose that Q is a nonempty subset of an ordered semiring $(\mathcal{S}, +, \cdot, \leq)$. Then Q is called an ordered quasi ideal of $\mathcal{S}$, if $(Q, +)$ is a subsemigroup of $(\mathcal{S}, +)$ and*
(a) $(\sum Q\mathcal{S}] \cap (\sum \mathcal{S}Q] \subseteq Q$.
(b) If $x \leq q$ for some $q \in Q$, then $x \in Q$ (i.e., $Q = (Q]$).

Definition 2.4. [8] *Suppose that B is a nonempty subset of an ordered semiring $(\mathcal{S}, +, \cdot, \leq)$. Then B is called an ordered bi ideal of $\mathcal{S}$, if $(B, +)$ is a subsemigroup of $(\mathcal{S}, +)$ and*
(a) $B\mathcal{S}B \subseteq B$.
(b) If $x \leq b$ for some $b \in B$, then $x \in B$ (i.e., $B = (B]$).

Remark 2.1. [7] *For any nonempty subsets A, B of an ordered semiring $\mathcal{S}$, we denote*
(i) $\sum A = \left\{ \sum_{i=1}^{n} a_i \in \mathcal{S} | a_i \in A, n \in \mathbb{N} \right\}$,
(ii) $\sum AB = \left\{ \sum_{i=1}^{n} a_i b_i \in \mathcal{S} | a_i \in A, b_i \in B, n \in \mathbb{N} \right\}$,
(iii) $(A] = \{x \in \mathcal{S} | x \leq a$ for some $a \in A\}$.

Remark 2.2. [7] *For any nonempty subsets A, B of an ordered semiring $\mathcal{S}$, then*
(i) $A \subseteq \sum A$ *and* $\sum(\sum A) = \sum A$.
(ii) $A \subseteq B$ *and* $\sum A \subseteq \sum B$.
(iii) $A(\sum B) \subseteq (\sum A)(\sum B) \subseteq \sum AB$ *and* $(\sum A)B \subseteq (\sum A)(\sum B) \subseteq \sum AB$.
(iv) $\sum(A] \subseteq (\sum A]$.
(v) $A \subseteq (A]$ *and* $((A]] = (A]$.
(vi) If $A \subseteq B$, *then* $(A] \subseteq (B]$.
(vii) $A(B] \subseteq (A](B] \subseteq (AB]$ *and* $(A]B \subseteq (A](B] \subseteq (AB]$.

Definition 2.5. [15] *Suppose that I, B and Q are nonempty subsets of a semiring $(S, +, \cdot)$. Then*
(i) I is called a right (left) $\mathcal{A}$-ideal of S if $IS \cap I \neq \emptyset$ $(SI \cap I \neq \emptyset)$.
(ii) I is called an $\mathcal{A}$-ideal of S if I is a right $\mathcal{A}$-ideal and left $\mathcal{A}$-ideal of S.
(iii) A subsemiring B of S is called a bi $\mathcal{A}$-ideal if $BSB \cap B \neq \emptyset$.
(iv) A subsemiring Q of S is called a quasi $\mathcal{A}$-ideal if $[QS \cap SQ] \cap Q \neq \emptyset$.
(v) A subsemiring Q is called a right (left) bi quasi $\mathcal{A}$-ideal of S if $[QS \cap QSQ] \cap Q \neq \emptyset$ $([SQ \cap QSQ] \cap Q \neq \emptyset)$.
(vi) Q is called a bi quasi $\mathcal{A}$-ideal of S if Q is a left bi quasi $\mathcal{A}$-ideal and right bi quasi $\mathcal{A}$-ideal of S.

Definition 2.6. [15] *Suppose that I and Q are nonempty subsets of a semiring $(S, +, \cdot)$. Then*
(i) I is called a right (left) tri $\mathcal{A}$-ideal of S if $I^2SI \cap I \neq \emptyset$ $\left(ISI^2 \cap I \neq \emptyset\right)$.
(ii) I is called a tri $\mathcal{A}$-ideal of S if I is a right tri $\mathcal{A}$-ideal and left tri $\mathcal{A}$-ideal of S.
(iii) Q is called a right (left) tri quasi $\mathcal{A}$-ideal of S if Q is a subsemiring of $\mathcal{S}$ and $\left[QS \cap Q^2SQ\right] \cap Q \neq \emptyset$ $\left(\left[SQ \cap QSQ^2\right] \cap Q \neq \emptyset\right)$.
(iv) Q is called a tri quasi $\mathcal{A}$-ideal of S if Q is a right tri quasi $\mathcal{A}$-ideal and left tri quasi $\mathcal{A}$-ideal of S.

3. **Ordered $\mathcal{A}$-Ideals.** Here $\mathcal{S}$ stands for additively commutative ordered semiring and $\mathscr{R}$ denotes non negative real numbers unless otherwise stated.

Definition 3.1. *Suppose that I is a nonempty subset of an ordered semiring $(\mathcal{S}, +, \cdot, \leq)$. Then I is called an ordered right (left) $\mathcal{A}$-ideal of $\mathcal{S}$, if $(I, +)$ is a subsemigroup of $(\mathcal{S}, +)$,*
(a) I is a right (left) $\mathcal{A}$-ideal of $\mathcal{S}$.
(b) If $x \leq i$ for some $i \in I$, then $x \in I$ (i.e., $I = (I]$). Hence, I is called an ordered $\mathcal{A}$-ideal if I is an ordered right $\mathcal{A}$-ideal and ordered left $\mathcal{A}$-ideal of $\mathcal{S}$.

Lemma 3.1. *Let I be a nonempty subset of $\mathcal{S}$. Then*
(i) $(\sum I\mathcal{S}]$ is an ordered right $\mathcal{A}$-ideal of $\mathcal{S}$.
(ii) $(\sum \mathcal{S}I]$ is an ordered left $\mathcal{A}$-ideal of $\mathcal{S}$.
(iii) $(\sum \mathcal{S}I\mathcal{S}]$ is an ordered $\mathcal{A}$-ideal of $\mathcal{S}$.

Proof: Let $x, y \in (\sum I\mathcal{S}]$. Then, $x \leq x'$, $y \leq y'$ for some $x', y' \in \sum I\mathcal{S}$. Clearly, $x + y \leq x' + y'$ and $x' + y' \in \sum I\mathcal{S}$ implies that $x + y \in (\sum I\mathcal{S}]$. Now, $(\sum I\mathcal{S}]\mathcal{S} \cap (\sum I\mathcal{S}] \subseteq (\sum I\mathcal{S}\mathcal{S}] \cap (\sum I\mathcal{S}] \subseteq (\sum I\mathcal{S}] \cap (\sum I\mathcal{S}] \neq \emptyset$. Also, $((\sum I\mathcal{S}]] = (\sum I\mathcal{S}]$. Hence, $(\sum I\mathcal{S}]$ is an ordered right $\mathcal{A}$-ideal of $\mathcal{S}$. It is similar to prove (ii) and (iii).

Definition 3.2. *Suppose that B and Q are nonempty subsets of $\mathcal{S}$. Then*
(1) B is called an ordered bi $\mathcal{A}$-ideal if $(B, +)$ is a subsemigroup of $(\mathcal{S}, +)$, $B\mathcal{S}B \cap B \neq \emptyset$ and $B = (B]$. It is equivalent to $(\sum B\mathcal{S}B] \cap B \neq \emptyset$.
(2) Q is called an ordered quasi $\mathcal{A}$-ideal if $(Q, +)$ is a subsemigroup of $(\mathcal{S}, +)$, $[(\sum Q\mathcal{S}] \cap (\sum \mathcal{S}Q]] \cap Q \neq \emptyset$ and $Q = (Q]$.

Definition 3.3. *Suppose that Q is a nonempty subset of $\mathcal{S}$. Then*
(1) Q is called an ordered right (left) bi quasi ideal of $\mathcal{S}$ if $(Q, +)$ is a subsemigroup of

$(\mathcal{S},+)$, $(\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}Q] \subseteq Q$ $((\sum \mathcal{S}Q] \cap (\sum Q\mathcal{S}Q] \subseteq Q)$ and $Q = (Q]$.
(2) Q is called an ordered bi quasi ideal of $\mathcal{S}$ if Q is an ordered right bi quasi ideal and ordered left bi quasi ideal of $\mathcal{S}$.

Definition 3.4. *Suppose that Q is a nonempty subset of $\mathcal{S}$. Then*
(1) Q is called an ordered right (left) bi quasi $\mathcal{A}$-ideal of $\mathcal{S}$ if $(Q,+)$ is a subsemigroup of $(\mathcal{S},+)$, $[(\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}Q]] \cap Q \neq \emptyset$ $([(\sum \mathcal{S}Q] \cap (\sum Q\mathcal{S}Q]] \cap Q \neq \emptyset)$ and $Q = (Q]$.
(2) Q is called an ordered bi quasi $\mathcal{A}$-ideal of $\mathcal{S}$ if Q is an ordered left bi quasi $\mathcal{A}$-ideal and ordered right bi quasi $\mathcal{A}$-ideal of $\mathcal{S}$.

Theorem 3.1. *Every ordered ideal (bi ideal, quasi ideal) is an ordered $\mathcal{A}$-ideal (bi $\mathcal{A}$-ideal, quasi $\mathcal{A}$-ideal).*

Proof: Suppose that I is an ordered ideal of $\mathcal{S}$. Now, $(\sum I\mathcal{S}] \cap I \subseteq I \cap I \neq \emptyset$ and $(\sum \mathcal{S}I] \cap I \subseteq I \cap I \neq \emptyset$. Hence, I is an ordered $\mathcal{A}$-ideal of $\mathcal{S}$.

Converse of Theorem 3.1 may not be true by the following counter example.

Example 3.1. *Consider the semiring $\mathcal{S}_1 = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ with the following compositions:*

$+$	s_1	s_2	s_3	s_4	s_5	s_6
s_1	s_1	s_2	s_3	s_4	s_5	s_6
s_2	s_2	s_2	s_3	s_4	s_5	s_6
s_3	s_3	s_3	s_3	s_6	s_5	s_6
s_4	s_4	s_4	s_6	s_4	s_5	s_6
s_5	s_5	s_5	s_5	s_5	s_5	s_5
s_6	s_6	s_6	s_6	s_6	s_5	s_6

$\cdot$	s_1	s_2	s_3	s_4	s_5	s_6
s_1	s_1	s_1	s_1	s_1	s_1	s_1
s_2	s_1	s_2	s_3	s_2	s_3	s_3
s_3	s_1	s_2	s_3	s_2	s_3	s_3
s_4	s_1	s_4	s_5	s_4	s_5	s_5
s_5	s_1	s_4	s_5	s_4	s_5	s_5
s_6	s_1	s_4	s_5	s_4	s_5	s_5

Define a binary relation $\leq$ on $\mathcal{S}_1$ by $\leq := \{(x,x) | x \in \mathcal{S}_1\}$. Then $(\mathcal{S}_1, +, \cdot, \leq)$ is an additively commutative ordered semiring. (i) Let $I = \{s_1, s_2\}$, $I+I \subseteq I$ and $I = (I]$. Clearly, I is an ordered $\mathcal{A}$-ideal of $\mathcal{S}_1$ but I is not an ordered ideal of $\mathcal{S}_1$ by $(\sum I\mathcal{S}_1] = \{s_1, s_2, s_3\} \nsubseteq I$ and $(\sum \mathcal{S}_1 I] = \{s_1, s_2, s_4\} \nsubseteq I$. (ii) Let $Q = \{s_2, s_3\}$, $Q+Q \subseteq Q$ and $Q = (Q]$. Clearly, Q is an ordered quasi $\mathcal{A}$-ideal of $\mathcal{S}_1$ by $[(\sum Q\mathcal{S}_1] \cap (\sum \mathcal{S}_1 Q]] \cap Q = [\{s_1, s_2, s_3\} \cap \mathcal{S}_1] \cap \{s_2, s_3\} = Q \neq \emptyset$. However, Q is not an ordered quasi ideal of $\mathcal{S}_1$ by $(\sum Q\mathcal{S}_1] \cap (\sum \mathcal{S}_1 Q] = \{s_1, s_2, s_3\} \nsubseteq Q$. (iii) Let $B = \{s_4, s_5\}$, $B+B \subseteq B$ and $B = (B]$. Clearly, B is an ordered bi $\mathcal{A}$-ideal of $\mathcal{S}_1$ by $(\sum B\mathcal{S}_1 B] \cap B = \{s_1, s_4, s_5\} \cap \{s_4, s_5\} = B \neq \emptyset$. However, B is not an ordered bi ideal of $\mathcal{S}_1$ by $(\sum B\mathcal{S}_1 B] = \{s_1, s_4, s_5\} \nsubseteq B$.

Example 3.2. *Let $\mathcal{S}_2 = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & a_6 & a_7 & a_8 & a_9 \\ 0 & 0 & 0 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 & 0 & a_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, a_i'^s \in \mathscr{R} \right\}$ be an ordered semiring under the ordinary multiplication of real numbers with partial ordered relation $\leq$. Now, we define partial order relation $\leq$ on $\mathcal{S}_2$, for any $A, B \in \mathcal{S}_2$, $A \leq_{S_2} B$ if and only if $a_{ij} \leq_N b_{ij}$, for all i and j. Then it is easy to verify that $\mathcal{S}_2$ is an ordered semiring under usual multiplication of matrices over real numbers $\mathscr{R}$ with partial order relation $\leq$. Clearly, $B = \left\{ \begin{pmatrix} 0 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, b_i'^s \in \mathscr{R} \right\}$ is an ordered bi $\mathcal{A}$-ideal of $\mathcal{S}_2$, but B is not an ordered bi ideal of $\mathcal{S}_2$ by $(\sum B\mathcal{S}_2 B] = \left\{ \begin{pmatrix} 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, c_i'^s \in \mathscr{R} \right\} \nsubseteq B$.*

Theorem 3.2. *Every ordered quasi ideal (bi ideal) is an ordered bi quasi ideal.*

Proof: Suppose that Q is an ordered quasi ideal of $\mathcal{S}$. Now, $(\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}Q] \subseteq (\sum Q\mathcal{S}] \cap (\sum \mathcal{S}\mathcal{S}Q] \subseteq (\sum Q\mathcal{S}] \cap (\sum \mathcal{S}Q] \subseteq Q$ and $(\sum \mathcal{S}Q] \cap (\sum Q\mathcal{S}Q] \subseteq (\sum Q\mathcal{S}] \cap (\sum \mathcal{S}Q] \subseteq Q$. Hence, Q is an ordered bi quasi ideal of $\mathcal{S}$.

Converse of Theorem 3.2 may not be true by the following example.

Example 3.3. *In Example 3.2, $\mathcal{S}_2$ is not regular by* $a = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \end{pmatrix} \in \mathcal{S}_2$ *there is no* $x \in \mathcal{S}_2$ *such that* $a \leq axa$. *Let* $Q = \left\{ \begin{pmatrix} 0&0&x_1&0&0&x_2 \\ 0&0&x_3&0&x_4&x_5 \\ 0&0&0&0&0&x_6 \\ 0&0&0&0&0&x_7 \\ 0&0&0&0&0&x_8 \\ 0&0&0&0&0&0 \end{pmatrix} \middle| x_i^{\prime s} \in \mathscr{R} \right\}$. *Now,* $(\sum Q\mathcal{S}_2] = \left\{ \begin{pmatrix} 0&0&0&y_1&y_2&y_3 \\ 0&0&0&y_4&y_5&y_6 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \middle| y_i^{\prime s} \in \mathscr{R} \right\}$, $(\sum \mathcal{S}_2 Q] = \left\{ \begin{pmatrix} 0&0&z_1&0&z_2&z_3 \\ 0&0&0&0&0&z_4 \\ 0&0&0&0&0&z_5 \\ 0&0&0&0&0&z_6 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \middle| z_i^{\prime s} \in \mathscr{R} \right\}$. *Thus,* $(\sum \mathcal{S}_2 Q] \cap (\sum Q\mathcal{S}_2 Q] = \left\{ \begin{pmatrix} 0&0&0&0&0&u_1 \\ 0&0&0&0&0&u_2 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \middle| u_i^{\prime s} \in \mathscr{R} \right\} \subseteq Q$, $(\sum Q\mathcal{S}_2] \cap (\sum Q\mathcal{S}_2 Q] = \left\{ \begin{pmatrix} 0&0&0&0&0&v_1 \\ 0&0&0&0&0&v_2 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \middle| v_i^{\prime s} \in \mathscr{R} \right\} \subseteq Q$. *Hence, Q is an ordered bi quasi ideal but Q is not an ordered quasi ideal of $\mathcal{S}_2$ by* $(\sum Q\mathcal{S}_2] \cap (\sum \mathcal{S}_2 Q] = \left\{ \begin{pmatrix} 0&0&0&0&w_1&w_2 \\ 0&0&0&0&0&w_3 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \middle| w_i^{\prime s} \in \mathscr{R} \right\} \nsubseteq Q$.

Corollary 3.1. *Every ordered bi quasi $\mathcal{A}$-ideal is an ordered quasi $\mathcal{A}$-ideal.*

Proof: Suppose that Q is an ordered bi quasi $\mathcal{A}$-ideal of $\mathcal{S}$. Now, $\emptyset \neq [(\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}Q]] \cap Q \subseteq (\sum Q\mathcal{S}] \cap Q$ and $\emptyset \neq [(\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}Q]] \cap Q \subseteq (\sum Q\mathcal{S}Q] \cap Q \subseteq (\sum \mathcal{S}\mathcal{S}Q] \cap Q \subseteq (\sum \mathcal{S}Q] \cap Q$. Thus, $\emptyset \neq [(\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}Q]] \cap Q \subseteq [(\sum Q\mathcal{S}] \cap (\sum \mathcal{S}Q]] \cap Q$. Hence, Q is an ordered quasi $\mathcal{A}$-ideal of $\mathcal{S}$.

Converse of Corollary 3.1 is not true by the following example.

Example 3.4. *The ordered semiring* $\mathcal{S}_3 = \left\{ \begin{pmatrix} 0&r_1&r_2&r_3 \\ 0&0&r_4&r_5 \\ 0&0&0&r_6 \\ 0&0&0&r_7 \end{pmatrix} \middle| r_i^{\prime s} \in \mathscr{R} \right\}$ *is not regular. Let* $Q = \left\{ \begin{pmatrix} 0&0&x_1&0 \\ 0&0&0&x_2 \\ 0&0&0&0 \\ 0&0&0&x_3 \end{pmatrix} \middle| x_i^{\prime s} \in \mathscr{R} \right\}$ *be an ordered quasi $\mathcal{A}$-ideal of $\mathcal{S}_3$ but Q is not an ordered bi quasi $\mathcal{A}$-ideal of $\mathcal{S}_3$ by* $\left[\left(\sum_{i=1}^{n} r_i' q\right] \cap \left(\sum_{i=1}^{n} q r_i'' q\right]\right] \cap q = \emptyset$ *and* $\left[\left(\sum_{i=1}^{n} q r_i'\right] \cap \left(\sum_{i=1}^{n} q r_i'' q\right]\right] \cap q = \emptyset$ *with the $(n-1)$ q terms as zero, where* $q = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&1 \end{pmatrix} \in Q$ *and* $r_i' = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&1 \\ 0&0&0&1 \\ 0&0&0&1 \end{pmatrix} \in \mathcal{S}_3$ *and* $r_i'' = \begin{pmatrix} 0&1&1&1 \\ 0&0&1&1 \\ 0&0&0&1 \\ 0&0&0&1 \end{pmatrix} \in \mathcal{S}_3$.

Theorem 3.3. *Every ordered bi $\mathcal{A}$-ideal is an ordered quasi $\mathcal{A}$-ideal.*

Proof: Suppose that B is an ordered bi $\mathcal{A}$-ideal of $\mathcal{S}$. Now, $\emptyset \neq (\sum B\mathcal{S}B] \cap B \subseteq (\sum B\mathcal{S}] \cap B$ and $\emptyset \neq (\sum B\mathcal{S}B] \cap B \subseteq (\sum \mathcal{S}B] \cap B$. Thus, $\emptyset \neq (\sum B\mathcal{S}B] \cap B \subseteq [(\sum B\mathcal{S}] \cap (\sum \mathcal{S}B]] \cap B$. Hence, B is an ordered quasi $\mathcal{A}$-ideal of $\mathcal{S}$.

Converse of Theorem 3.3 may not be true by the following counter example.

Example 3.5. *Let* $B = \left\{ \begin{pmatrix} 0&0&0&x_1 \\ 0&0&x_2&x_3 \\ 0&0&0&0 \\ 0&0&0&x_4 \end{pmatrix} \middle| x_i^{\prime s} \in \mathscr{R} \right\} \subseteq \mathcal{S}_3$ *in Example 3.4. Now,* $(\sum \mathcal{S}_3 B] = \left\{ \begin{pmatrix} 0&0&y_1&y_2 \\ 0&0&0&y_3 \\ 0&0&0&y_4 \\ 0&0&0&y_5 \end{pmatrix} \middle| y_i^{\prime s} \in \mathscr{R} \right\}$ *and* $(\sum B\mathcal{S}_3] = \left\{ \begin{pmatrix} 0&0&0&z_1 \\ 0&0&0&z_2 \\ 0&0&0&0 \\ 0&0&0&z_3 \end{pmatrix} \middle| z_i^{\prime s} \in \mathscr{R} \right\}$. *Hence,* $[(\sum B\mathcal{S}_3] \cap (\sum \mathcal{S}_3 B]] \cap B \neq \emptyset$. *Thus, B is an ordered quasi $\mathcal{A}$-ideal of $\mathcal{S}_3$ but B is not an ordered bi $\mathcal{A}$-ideal of $\mathcal{S}_3$ by* $\left(\sum_{i=1}^{n} b r_i b\right] \cap b = \emptyset$ *with the $(n-1)$ b terms as zero, where* $b = \begin{pmatrix} 0&0&0&0 \\ 0&0&1&1 \\ 0&0&0&0 \\ 0&0&0&1 \end{pmatrix} \in B$ *and* $r' = \begin{pmatrix} 0&1&1&1 \\ 0&0&1&1 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix} \in \mathcal{S}_3$.

Theorem 3.4. *Every ordered quasi $\mathcal{A}$-ideal is an ordered $\mathcal{A}$-ideal.*

Proof: Suppose that Q is an ordered quasi $\mathcal{A}$-ideal of $\mathcal{S}$, then $[(\sum Q\mathcal{S}]\cap(\sum \mathcal{S}Q]]\cap Q \neq \emptyset$. Now, $\emptyset \neq [(\sum Q\mathcal{S}]\cap(\sum \mathcal{S}Q]]\cap Q \subseteq (\sum Q\mathcal{S}]\cap Q$ and $\emptyset \neq [(\sum Q\mathcal{S}]\cap(\sum \mathcal{S}Q]]\cap Q \subseteq (\sum \mathcal{S}Q]\cap Q$. Hence, Q is an ordered $\mathcal{A}$-ideal of $\mathcal{S}$.

Converse of Theorem 3.4 is not true by the following example.

Example 3.6. *The ordered semiring* $\mathcal{S}_4 = \left\{\left.\begin{pmatrix} 0 & r_1 & r_2 & r_3 \\ 0 & 0 & r_4 & r_5 \\ 0 & 0 & 0 & r_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right| r_i'^s \in \mathscr{R}\right\}$ *is not regular. Let* $Q = \left\{\left.\begin{pmatrix} 0 & 0 & q_1 & 0 \\ 0 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right| q_i'^s \in \mathscr{R}\right\}$ *be an ordered $\mathcal{A}$-ideal but Q is not an ordered quasi $\mathcal{A}$-ideal of $\mathcal{S}_4$ by* $\left[\left(\sum q r_i'\right]\cap\left(\sum r_i'' q\right]\right]\cap q = \emptyset$ *with some $(n-1) q_i'^s$ as zero, where* $q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in Q$, $r_i' = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}_4$ *and* $r_i'' = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}_4$.

Theorem 3.5. *Every ordered bi quasi ideal is an ordered bi quasi $\mathcal{A}$-ideal.*

Proof: Suppose that Q is an ordered bi quasi ideal of $\mathcal{S}$, then $[(\sum Q\mathcal{S}]\cap(\sum Q\mathcal{S}Q]] \subseteq Q$ and $[(\sum \mathcal{S}Q]\cap(\sum Q\mathcal{S}Q]] \subseteq Q$. Now, $[(\sum Q\mathcal{S}]\cap(\sum Q\mathcal{S}Q]]\cap Q \subseteq Q\cap Q \neq \emptyset$ and $[(\sum \mathcal{S}Q]\cap(\sum Q\mathcal{S}Q]]\cap Q \subseteq Q\cap Q \neq \emptyset$. Hence, Q is an ordered bi quasi $\mathcal{A}$-ideal of $\mathcal{S}$.

Converse of Theorem 3.5 is not true by the example.

Example 3.7. *Consider the ordered semiring* $\mathcal{S}_5 = \left\{\left.\begin{pmatrix} 0 & 0 & r_2 & r_3 \\ 0 & 0 & r_4 & r_5 \\ 0 & 0 & 0 & r_6 \\ 0 & 0 & 0 & r_7 \end{pmatrix}\right| r_i'^s \in \mathscr{R}\right\}$. *Let* $Q = \left\{\left.\begin{pmatrix} 0 & 0 & x_1 & 0 \\ 0 & 0 & x_2 & x_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix}\right| x_i'^s \in \mathscr{R}\right\}$. *Now,* $(\sum \mathcal{S}_5 Q] = \left\{\left.\begin{pmatrix} 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & y_4 \\ 0 & 0 & 0 & y_5 \end{pmatrix}\right| y_i'^s \in \mathscr{R}\right\}$ *and* $(\sum Q\mathcal{S}_5] = \left\{\left.\begin{pmatrix} 0 & 0 & 0 & z_1 \\ 0 & 0 & 0 & z_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_3 \end{pmatrix}\right| z_i'^s \in \mathscr{R}\right\}$ *and* $(\sum Q\mathcal{S}_5 Q] = \left\{\left.\begin{pmatrix} 0 & 0 & 0 & l_1 \\ 0 & 0 & 0 & l_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_3 \end{pmatrix}\right| l_i'^s \in \mathscr{R}\right\}$. *Thus,* $[(\sum \mathcal{S}_5 Q]\cap(\sum Q\mathcal{S}_5 Q]]\cap Q = \left\{\left.\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2 \end{pmatrix}\right| u_i'^s \in \mathscr{R}\right\} \neq \emptyset$ *and* $[(\sum Q\mathcal{S}_5]\cap(\sum Q\mathcal{S}_5 Q]]\cap Q = \left\{\left.\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2 \end{pmatrix}\right| v_i'^s \in \mathscr{R}\right\} \neq \emptyset$. *Hence, Q is an ordered bi quasi $\mathcal{A}$-ideal but Q is not an ordered bi quasi ideal of $\mathcal{S}_5$ by* $(\sum Q\mathcal{S}_5]\cap(\sum Q\mathcal{S}_5 Q] = \left\{\left.\begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \end{pmatrix}\right| x, y, z \in \mathscr{R}\right\} \nsubseteq Q$.

Theorem 3.6. *If Q is an ordered $\mathcal{A}$-ideal (bi $\mathcal{A}$-ideal, quasi $\mathcal{A}$-ideal, bi quasi $\mathcal{A}$-ideal) of $\mathcal{S}$ and $Q \subseteq Q' \subseteq \mathcal{S}$, then Q' is an ordered $\mathcal{A}$-ideal (bi $\mathcal{A}$-ideal, quasi $\mathcal{A}$-ideal, bi quasi $\mathcal{A}$-ideal) of $\mathcal{S}$.*

Proof: Suppose that Q is an ordered bi quasi $\mathcal{A}$ ideal of $\mathcal{S}$ with $Q \subseteq Q' \subseteq \mathcal{S}$. Then $\emptyset \neq [(\sum Q\mathcal{S}]\cap(\sum Q\mathcal{S}Q]]\cap Q \subseteq [(\sum Q'\mathcal{S}]\cap(\sum Q'\mathcal{S}Q']]\cap Q'$ and $\emptyset \neq [(\sum \mathcal{S}Q]\cap(\sum Q\mathcal{S}Q]]\cap Q \subseteq [(\sum \mathcal{S}Q']\cap(\sum Q'\mathcal{S}Q']]\cap Q'$. Therefore, Q' is an ordered bi quasi $\mathcal{A}$ ideal of $\mathcal{S}$.

Corollary 3.2. *The union of ordered $\mathcal{A}$-ideals (bi $\mathcal{A}$-ideals, quasi $\mathcal{A}$-ideals, bi quasi $\mathcal{A}$-ideals) of $\mathcal{S}$ is an ordered $\mathcal{A}$-ideal (bi $\mathcal{A}$-ideal, quasi $\mathcal{A}$-ideal, bi quasi $\mathcal{A}$-ideal) of $\mathcal{S}$.*

Proof: Let I_1 and I_2 be any two ordered $\mathcal{A}$-ideals of $\mathcal{S}$. Then $I_1 \subseteq I_1 \cup I_2$, by Theorem 3.6, $I_1 \cup I_2$ is an ordered $\mathcal{A}$-ideal of $\mathcal{S}$.

4. **Ordered Tri $\mathcal{A}$-Ideals.**

Definition 4.1. *Suppose that I is a nonempty subset of $\mathcal{S}$. Then I is called an ordered right (left) tri $\mathcal{A}$-ideal of $\mathcal{S}$, if $(I,+)$ is a subsemigroup of $(\mathcal{S},+)$ and*
(a) I is a right (left) tri $\mathcal{A}$-ideal of $\mathcal{S}$.
(b) If $x \leq i$ for some $i \in I$, then $x \in I$ (i.e., $I = (I]$).

I is called an ordered tri $\mathcal{A}$-ideal if I is an ordered right tri $\mathcal{A}$-ideal and ordered left tri $\mathcal{A}$-ideal of $\mathcal{S}$.

Lemma 4.1. *Let I be a nonempty subset of $\mathcal{S}$. Then*
(i) $\left(\sum I^2\mathcal{S}I\right]$ is an ordered right tri $\mathcal{A}$-ideal of $\mathcal{S}$.
(ii) $\left(\sum I\mathcal{S}I^2\right]$ is an ordered left tri $\mathcal{A}$-ideal of $\mathcal{S}$.

Proof: Let $x, y \in \left(\sum I^2\mathcal{S}I\right]$. Then, $x \leq x'$, $y \leq y'$ for some $x', y' \in \sum I^2\mathcal{S}I$. Clearly, $x + y \leq x' + y'$ and $x' + y' \in \sum I^2\mathcal{S}I$ implies that $x + y \in \left(\sum I^2\mathcal{S}I\right]$. Now, $\left[\left(\left(\sum I^2\mathcal{S}I\right]\right)^2\mathcal{S}\left(\sum I^2\mathcal{S}I\right]\right] \cap \left(\sum I^2\mathcal{S}I\right] \subseteq \left(\sum I^2\mathcal{S}II^2\mathcal{S}II\mathcal{S}I^2\mathcal{S}I\right] \cap \left(\sum I^2\mathcal{S}I\right] \subseteq \left(\sum I^2\mathcal{S}I\right] \cap \left(\sum I^2\mathcal{S}I\right] \neq \emptyset$. Also, $\left(\left(\sum I^2\mathcal{S}I\right]\right] = \left(\sum I^2\mathcal{S}I\right]$. Hence, $\left(\sum I^2\mathcal{S}I\right]$ is an ordered right tri $\mathcal{A}$-ideal of $\mathcal{S}$. It is similar to prove (ii).

Definition 4.2. *Suppose that Q is a nonempty subset of $\mathcal{S}$. Then*
(i) Q is called an ordered right (left) tri quasi ideal of $\mathcal{S}$ if $(Q, +)$ is a subsemigroup of $(\mathcal{S}, +)$ and $(\sum Q\mathcal{S}] \cap \left(\sum Q^2\mathcal{S}Q\right] \subseteq Q$ $\left((\sum \mathcal{S}Q] \cap \left(\sum Q\mathcal{S}Q^2\right] \subseteq Q\right)$.
(ii) Q is called an ordered tri quasi ideal of $\mathcal{S}$ if Q is an ordered right tri quasi ideal and ordered left tri quasi ideal of $\mathcal{S}$.

Definition 4.3. *Suppose that Q is a nonempty subset of $\mathcal{S}$. Then*
(i) Q is called an ordered right (left) tri quasi $\mathcal{A}$-ideal of $\mathcal{S}$ if $(Q, +)$ is a subsemigroup of $(\mathcal{S}, +)$ and $\left[(\sum Q\mathcal{S}] \cap \left(\sum Q^2\mathcal{S}Q\right]\right] \cap Q \neq \emptyset$ $\left(\left[(\sum \mathcal{S}Q] \cap \left(\sum Q^2\mathcal{S}Q\right]\right] \cap Q \neq \emptyset\right)$.
(ii) Q is called an ordered tri quasi $\mathcal{A}$-ideal of $\mathcal{S}$ if Q is an ordered right tri quasi $\mathcal{A}$-ideal and ordered left tri quasi $\mathcal{A}$-ideal of $\mathcal{S}$.

Theorem 4.1. *Every ordered tri $\mathcal{A}$-ideal is an ordered $\mathcal{A}$-ideal (ordered bi $\mathcal{A}$-ideal).*

Proof: Suppose that I is an ordered tri $\mathcal{A}$-ideal of $\mathcal{S}$, then $(I, +)$ is a subsemigroup of $(\mathcal{S}, +)$ and $\left(\sum I^2\mathcal{S}I\right] \cap I \neq \emptyset$ and $\left(\sum I\mathcal{S}I^2\right] \cap I \neq \emptyset$. Now, $\emptyset \neq \left(\sum I^2\mathcal{S}I\right] \cap I \subseteq (\sum I\mathcal{S}\mathcal{S}\mathcal{S}] \cap I \subseteq (\sum I\mathcal{S}] \cap I$ and $\emptyset \neq \left(\sum I\mathcal{S}I^2\right] \cap I \subseteq (\sum \mathcal{S}\mathcal{S}\mathcal{S}I] \cap I \subseteq (\sum \mathcal{S}I] \cap I$. Hence, I is an ordered $\mathcal{A}$-ideal of $\mathcal{S}$.

Converse of Theorem 4.1 may not be true by the example.

Example 4.1. *Let $\mathcal{S}_1 = \left\{\left.\begin{pmatrix} 0 & r_1 & r_2 & r_3 \\ 0 & 0 & r_4 & r_5 \\ 0 & 0 & 0 & r_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right| r_i'^s \in \mathscr{R}\right\}$ be an ordered semiring and not regular. Let $I = \left\{\left.\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i_1 & 0 \\ 0 & 0 & 0 & i_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right| i_i'^s \in \mathscr{R}\right\}$. Clearly, I is an ordered $\mathcal{A}$-ideal of $\mathcal{S}_1$ but I is not an ordered tri $\mathcal{A}$-ideal of $\mathcal{S}_1$ by $\left(\sum_{i=1}^n x_i^2 r_i x_i\right] \cap x = \emptyset$ and $\left(\sum_{i=1}^n x_i r_i x_i^2\right] \cap x = \emptyset$ with the $(n-1)$ terms of x'^s and r'^s as zero. This implies that $x^2 r_1 x \cap x = \emptyset$ and $x r_1 x^2 \cap x = \emptyset$, where $x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in I$ and $r_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}_1$.*

Example 4.2. *Let $\mathcal{S}_2 = \left\{\left.\begin{pmatrix} 0 & r_1 & r_2 & r_3 & r_4 & r_5 \\ 0 & 0 & r_6 & r_7 & r_8 & r_9 \\ 0 & 0 & 0 & r_{10} & r_{11} & r_{12} \\ 0 & 0 & 0 & 0 & r_{13} & r_{14} \\ 0 & 0 & 0 & 0 & 0 & r_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right| r_i'^s \in \mathscr{R}\right\}$ be an ordered semiring and not regular. Clearly, $B = \left\{\left.\begin{pmatrix} 0 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right| b_i'^s \in \mathscr{R}\right\}$ is an ordered bi $\mathcal{A}$-ideal but B is not an ordered tri $\mathcal{A}$-ideal of $\mathcal{S}_2$ by $\left(\sum b_i r_i b_i^2\right] \cap b = \emptyset$ and $\left(\sum b_i^2 r_i b_i\right] \cap b = \emptyset$ with $(n-1)$ terms of b'^s and r'^s as zero. This implies that $brb^2 \cap b = \emptyset$ and $b^2 rb \cap b = \emptyset$, where $b = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in B$ and $r = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}_2$.*

Theorem 4.2. *Every ordered tri $\mathcal{A}$-ideal is an ordered quasi $\mathcal{A}$-ideal.*

Proof: Suppose that Q is an ordered tri $\mathcal{A}$-ideal of $\mathcal{S}$. Now, $\emptyset \neq \left(\sum Q^2\mathcal{S}Q\right] \cap Q \subseteq (\sum Q\mathcal{S}\mathcal{S}Q] \cap Q \subseteq (\sum Q\mathcal{S}] \cap Q$ and $\emptyset \neq \left(\sum Q^2\mathcal{S}Q\right] \cap Q \subseteq (\sum Q\mathcal{S}Q] \cap Q \subseteq (\sum \mathcal{S}Q] \cap Q$. Hence, $\emptyset \neq \left(\sum Q^2\mathcal{S}Q\right] \cap Q \subseteq [(\sum Q\mathcal{S}] \cap (\sum \mathcal{S}Q]] \cap Q$. Similarly, $\emptyset \neq \left(\sum Q\mathcal{S}Q^2\right] \cap Q \subseteq [(\sum Q\mathcal{S}] \cap (\sum \mathcal{S}Q]] \cap Q$. Hence, Q is an ordered quasi $\mathcal{A}$-ideal of $\mathcal{S}$.

Converse of Theorem 4.2 may not be true in the given example.

Example 4.3. *Consider $\mathcal{S}_2$ in Example 4.2,* $Q = \left\{ \left(\begin{smallmatrix} 0&0&0&q_1&0&0 \\ 0&0&0&0&q_2&0 \\ 0&0&0&0&0&q_3 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{smallmatrix} \right) \middle| \, q_i'^s \in \mathscr{R} \right\}$ *is an ordered quasi $\mathcal{A}$-ideal but Q is not an ordered tri $\mathcal{A}$-ideal of $\mathcal{S}_2$ by* $\left(\sum q_i^2 r_i q_i\right] \cap q = \emptyset$ *and* $\left(\sum q_i r_i q_i^2\right] \cap q = \emptyset$ *with $(n-1)$ terms of q'^s and r'^s are zero. This implies that* $q^2rq \cap q = \emptyset$ *and* $qrq^2 \cap q = \emptyset$, *where* $q = \left(\begin{smallmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{smallmatrix} \right) \in Q$ *and* $r = \left(\begin{smallmatrix} 0&1&1&1&1&1 \\ 0&0&1&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \end{smallmatrix} \right) \in \mathcal{S}_2$.

Corollary 4.1. *Every ordered tri $\mathcal{A}$-ideal is an ordered bi quasi $\mathcal{A}$-ideal.*

Proof: Suppose that Q is an ordered right tri $\mathcal{A}$-ideal of $\mathcal{S}$, then $\left(\sum Q^2\mathcal{S}Q\right] \cap Q \neq \emptyset$. Now, $\emptyset \neq \left(\sum Q^2\mathcal{S}Q\right] \cap Q \subseteq (\sum Q\mathcal{S}] \cap Q$ and $\emptyset \neq \left(\sum Q^2\mathcal{S}Q\right] \cap Q \subseteq (\sum Q\mathcal{S}Q] \cap Q$. This implies that $\emptyset \neq \left(\sum Q^2\mathcal{S}Q\right] \cap Q \subseteq [(\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}Q]] \cap Q$. Thus, Q is an ordered right bi quasi $\mathcal{A}$-ideal of $\mathcal{S}$. Suppose that Q is an ordered left tri $\mathcal{A}$-ideal of $\mathcal{S}$, then Q is an ordered left bi quasi $\mathcal{A}$-ideal of $\mathcal{S}$. Hence, Q is an ordered bi quasi $\mathcal{A}$-ideal of $\mathcal{S}$.

Converse of Corollary 4.1 may not be true in the given example.

Example 4.4. *Let* $\mathcal{S}_3 = \left\{ \left(\begin{smallmatrix} 0&0&0&0 \\ r_1&0&0&0 \\ r_2&r_3&0&0 \\ r_4&r_5&r_6&r_7 \end{smallmatrix} \right) \middle| \, r_i'^s \in \mathscr{R} \right\}$ *be an ordered semiring and $\mathcal{S}_3$ is not regular. Let* $Q = \left\{ \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ q_1&0&0&0 \\ 0&q_2&0&q_3 \end{smallmatrix} \right) \middle| \, q_i'^s \in \mathscr{R} \right\}$. *Now,* $(\sum \mathcal{S}_3Q] = \left\{ \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ c_1&c_2&0&c_3 \end{smallmatrix} \right) \middle| \, c_i'^s \in \mathscr{R} \right\}$ *and* $(\sum Q\mathcal{S}_3] = \left\{ \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ d_1&d_2&d_3&d_4 \end{smallmatrix} \right) \middle| \, d_i'^s \in \mathscr{R} \right\}$ *and* $(\sum Q\mathcal{S}_3Q] = \left\{ \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ e_1&e_2&0&e_3 \end{smallmatrix} \right) \middle| \, e_i'^s \in \mathscr{R} \right\}$. *Thus,* $[Q\mathcal{S}_3 \cap Q\mathcal{S}_3Q] \cap Q \neq \emptyset$ *and* $[\mathcal{S}_3Q \cap Q\mathcal{S}_3Q] \cap Q \neq \emptyset$. *Hence, Q is an ordered bi quasi $\mathcal{A}$-ideal of $\mathcal{S}_3$. However, Q is not an ordered tri $\mathcal{A}$-ideal of $\mathcal{S}_3$ by* $\left(\sum_{i=1}^{n} q_i^2 r_i q_i\right] \cap q = \emptyset$ *and* $\left(\sum_{i=1}^{n} q_i r_i q_i^2\right] \cap q = \emptyset$ *with the $(n-1)$ terms of q^s and r^s as zero. This implies that* $q^2rq \cap q = \emptyset$ *and* $qrq^2 \cap q = \emptyset$, *where* $q = \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&1&0&1 \end{smallmatrix} \right) \in Q$ *and* $r = \left(\begin{smallmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 1&1&0&0 \\ 1&1&1&1 \end{smallmatrix} \right) \in \mathcal{S}_3$.

Theorem 4.3. *Every ordered bi quasi ideal is an ordered tri quasi ideal.*

Proof: Suppose that Q is an ordered bi quasi ideal of $\mathcal{S}$, then $(\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}Q] \subseteq Q$ and $(\sum \mathcal{S}Q] \cap (\sum Q\mathcal{S}Q] \subseteq Q$. Now, $(\sum \mathcal{S}Q] \cap \left(\sum Q\mathcal{S}Q^2\right] \subseteq (\sum \mathcal{S}Q] \cap (\sum Q\mathcal{S}\mathcal{S}Q] \subseteq (\sum \mathcal{S}Q] \cap (\sum Q\mathcal{S}Q] \subseteq Q$ and $(\sum Q\mathcal{S}] \cap \left(\sum Q^2\mathcal{S}Q\right] \subseteq (\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}\mathcal{S}Q] \subseteq (\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}Q] \subseteq Q$. Hence, Q is an ordered tri quasi ideal of $\mathcal{S}$.

Converse of Theorem 4.3 may not be true by the following example.

Example 4.5. *Consider $\mathcal{S}_3$ in Example 4.4,* $Q = \left\{ \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ a_1&0&0&0 \\ 0&0&0&a_2 \end{smallmatrix} \right) \middle| \, a_i'^s \in \mathscr{R} \right\}$. *Now,* $(\sum \mathcal{S}_3Q] = \left\{ \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ c_1&0&0&c_2 \end{smallmatrix} \right) \middle| \, c_i'^s \in \mathscr{R} \right\}$, $(\sum Q\mathcal{S}_3] = \left\{ \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ d_1&d_2&d_3&d_4 \end{smallmatrix} \right) \middle| \, d_i'^s \in \mathscr{R} \right\}$, $(\sum Q\mathcal{S}_3Q] = \left\{ \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ e_1&0&0&e_2 \end{smallmatrix} \right) \middle| \, e_i'^s \in \mathscr{R} \right\}$ *and* $\left(\sum Q\mathcal{S}_3Q^2\right] = \left\{ \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&f \end{smallmatrix} \right) \middle| \, f \in \mathscr{R} \right\}$. *Thus,* $(\sum \mathcal{S}_3Q] \cap \left(\sum Q\mathcal{S}_3Q^2\right] \subseteq Q$. *Hence, Q is an ordered left tri quasi ideal of $\mathcal{S}_3$ but Q is not a left bi quasi ideal by* $(\sum \mathcal{S}_3Q] \cap (\sum Q\mathcal{S}_3Q] = \left\{ \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ h_1&0&0&h_3 \end{smallmatrix} \right) \middle| \, h_i'^s \in \mathscr{R} \right\} \not\subseteq Q$.

Corollary 4.2. *Every ordered tri quasi $\mathcal{A}$-ideal is an ordered bi quasi $\mathcal{A}$-ideal.*

Proof: Suppose that Q is an ordered tri quasi $\mathcal{A}$-ideal of $\mathcal{S}$, then $\big[(\sum Q\mathcal{S}] \cap \big(\sum Q^2\mathcal{S}Q\big]\big] \cap Q \neq \emptyset$ and $[(\sum \mathcal{S}Q] \cap (\sum Q\mathcal{S}Q^2]] \cap Q \neq \emptyset$. Now, $\emptyset \neq \big[(\sum Q\mathcal{S}] \cap \big(\sum Q^2\mathcal{S}Q\big]\big] \cap Q \subseteq [(\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}\mathcal{S}Q]] \cap Q \subseteq [(\sum Q\mathcal{S}] \cap (\sum Q\mathcal{S}Q]] \cap Q$ and $\emptyset \neq \big[(\sum \mathcal{S}Q] \cap \big(\sum Q\mathcal{S}Q^2\big]\big] \cap Q \subseteq [(\sum \mathcal{S}Q] \cap (\sum Q\mathcal{S}\mathcal{S}Q]] \cap Q \subseteq [(\sum \mathcal{S}Q] \cap (\sum Q\mathcal{S}Q]] \cap Q$. Hence, Q is an ordered bi quasi $\mathcal{A}$-ideal of $\mathcal{S}$.

Converse of Corollary 4.2 is not true by the following example.

Example 4.6. *Consider $\mathcal{S}_3$ in Example 4.4, $Q = \left\{ \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & a_4 \end{smallmatrix} \right) \middle| a_i'^s \in \mathscr{R} \right\}$ is an ordered bi quasi $\mathcal{A}$-ideal of $\mathcal{S}_3$ but Q is not an ordered tri quasi $\mathcal{A}$-ideal of $\mathcal{S}_3$ by $[(\sum_{i=1}^n r_i' q_i] \cap \big(\sum_{i=1}^n q_i r_i'' q_i^2\big]\big] \cap q = \emptyset$ and $[(\sum q_i r_i'] \cap (\sum q_i^2 r_i'' q_i]] \cap q = \emptyset$ with $(n-1)$ terms of q'^s, r'^s as zero. This implies that $[r'q \cap qr''q^2] \cap q = \emptyset$ and $[qr' \cap q^2 r'' q] \cap q = \emptyset$, where $q = \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 1&1&0&1 \end{smallmatrix} \right) \in Q$, $r' = \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&1&0&0 \\ 1&1&1&1 \end{smallmatrix} \right) \in \mathcal{S}_3$ and $r'' = \left(\begin{smallmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 1&0&0&0 \\ 0&1&1&1 \end{smallmatrix} \right) \in \mathcal{S}_3$.*

Theorem 4.4. *If Q is an ordered tri $\mathcal{A}$-ideal (tri quasi $\mathcal{A}$-ideal) of $\mathcal{S}$ and $Q \subseteq Q' \subseteq \mathcal{S}$, then Q' is an ordered tri $\mathcal{A}$-ideal (tri quasi $\mathcal{A}$-ideal) of $\mathcal{S}$.*

Proof: Suppose that Q is an ordered tri quasi $\mathcal{A}$-ideal of $\mathcal{S}$ with $Q \subseteq Q' \subseteq \mathcal{S}$. Then $\emptyset \neq \big[(\sum Q\mathcal{S}] \cap \big(\sum Q\mathcal{S}Q^2\big]\big] \cap Q \subseteq [(\sum Q'\mathcal{S}] \cap (\sum Q'\mathcal{S}Q'Q']] \cap Q'$ and $\emptyset \neq \big[(\sum \mathcal{S}Q] \cap \big(\sum Q^2\mathcal{S}Q\big]\big] \cap Q \subseteq [(\sum \mathcal{S}Q'] \cap (\sum Q'Q'\mathcal{S}Q']] \cap Q'$. Therefore, Q' is an ordered tri quasi $\mathcal{A}$-ideal of $\mathcal{S}$.

Corollary 4.3. *The union of ordered tri $\mathcal{A}$-ideals (tri quasi $\mathcal{A}$-ideals) of $\mathcal{S}$ is an ordered tri $\mathcal{A}$-ideal (tri quasi $\mathcal{A}$-ideal) of $\mathcal{S}$.*

Proof: Let Q_1 and Q_2 be any two ordered tri $\mathcal{A}$-ideals of $\mathcal{S}$. Then $Q_1 \subseteq Q_1 \cup Q_2$, by Theorem 4.4, $Q_1 \cup Q_2$ is an ordered tri $\mathcal{A}$-ideal of $\mathcal{S}$.

5. **Conclusion.** In this article, various ordered almost ideals including ordered quasi $\mathcal{A}$-ideals, ordered bi quasi $\mathcal{A}$-ideals, ordered tri $\mathcal{A}$-ideals, and ordered tri quasi $\mathcal{A}$-ideals in ordered semirings, are introduced. We discussed the implications ordered ideals $\Longrightarrow$ ordered quasi ideals $\Longrightarrow$ ordered bi quasi ideals $\Longrightarrow$ ordered tri quasi ideals $\Longrightarrow$ ordered tri quasi $\mathcal{A}$-ideals $\Longrightarrow$ ordered bi quasi $\mathcal{A}$-ideals $\Longrightarrow$ ordered bi $\mathcal{A}$-ideals $\Longrightarrow$ ordered quasi $\mathcal{A}$-ideals $\Longrightarrow$ ordered $\mathcal{A}$-ideals. With instances given, the contrary implications are false. We plan to characterize other classes of ordered hyper semirings in the future using different hyper $\mathcal{A}$-ideals.

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