*ICIC Express Letters*
*Part B: Applications*
Volume **8**, Number **8**, August **2017**

pp. 1227–1232

# CONSTRUCTION OF EXACT SOLITARY SOLUTIONS OF THE MODIFIED KDV EQUATION BY ADOMIAN DECOMPOSITION METHOD

Ao Zhu and Chengyang Fan

School of Automation
Wuhan University of Technology
No. 122, Luoshi Road, Wuhan 430070, P. R. China
{ aozhu; cyfan }@whut.edu.cn

Received March 2017; accepted May 2017

Abstract. *In this paper, the modified KdV equation: $u_t + u^2u_x + u_{xxx} = 0$, which exhibits solitary solutions, will be investigated. Explicit exact solitary solutions of the equation are obtained via Adomian decomposition method and symbolic computation system, Maple.*
**Keywords:** Modified KdV equation, Solitary solutions, Adomian decomposition method

## 1. Introduction.

Since the discovery of the soliton in 1965 by Zabusky and Kruskal, a large new domain of mathematical physics has developed. It is generally called the soliton theory. Constructing exact solutions for nonlinear equations in mathematical physics has long been a major concern for both mathematicians and physicists. Many nonlinear PDE's appear in condensed matter, solid state physics, fluid mechanics, chemical kinetics, plasma physics, nonlinear optics, propagation of fluxions in Josephon, theory of turbulence, ocean dynamics, biophysics and star formation and many other fields.

Explicit analytical solutions to the nonlinear equations are of fundamental importance. Various methods have been proposed to find explicit solutions to nonlinear evolution equations. Among them there are Hirota's dependent variable transformation method [1], sine-cosine method [2], the tanh-sech method [3], homogeneous balance method [4], Painlevé expansion, improved tanh-function method [5], Exp-function method [6], the Bäcklund transform method [7] and so on. A common idea for all these methods is that the equation is reduced by employing the transforms into simple equation that can be solved directly. Unlike these methods, the nonlinear equations are solved easily without transforms by using Adomian decomposition method. The Adomian decomposition method [8-10] provides the solution in the form of a rapidly convergent series that may give rise to the exact solution. In 2002, Wazwaz [11,12] has successfully used the decomposition method to construct solitary solutions for many nonlinear equations. In recent years, many researchers [13-18] have also given the solutions for nonlinear or linear equations by using Adomian decomposition method.

In this paper we would like to extend Adomian decomposition method to seek exact solutions for the modified KdV equation:

$$u_t + u^2u_x + u_{xxx} = 0. \tag{1}$$

The rest of this paper is organized as follows. In Section 2, we present the concrete scheme of the decomposition method for the modified KdV equation. In Section 3, the method is used to seek compactons for the MKdV equation: $u_t + u^2u_x + u_{xxx} = 0$ with the given initial condition. In Section 4, numerical results for the modofied KdV equation have been given. We conclude the paper in the last section.

2. **The Adomian Decomposition Method for Solving (1).** Consider nonlinear differential equation:

$$Lu + Ru + Nu = 0, \tag{2}$$

where $L$ is easily invertible linear differential operator and $R$ is the reminder of the linear operator, and $Nu$ represents the nonlinear term. If we apply the inverse operator $L^{-1}$ to both sides of (2), then we can obtain

$$L^{-1}Lu = -L^{-1}(Ru) - L^{-1}(Nu). \tag{3}$$

As one special case of (2), (1) can be written as $Lu = -Nu$, where

$$L = \frac{\partial}{\partial t}, \quad Nu = u^2 u_x + u_{xxx}. \tag{4}$$

$L^{-1}$ is given by the following:

$$L^{-1}(\cdot) = \int_0^t (\cdot)dt. \tag{5}$$

Applying the integral operator $L^{-1}$ to both sides of (1), and using the given initial condition, we obtain

$$u(x,t) = f(x) - L^{-1}\left(u^2 u_x + u_{xxx}\right), \tag{6}$$

where $f(x)$ is the function that arises from the given initial condition that is assumed to be prescribed. Adomian decomposition method [8-10] decomposes the unknown function $u(x,t)$ by a series of components

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \tag{7}$$

where the components $u_0, u_1, u_2, \ldots$ are usually determined recursively. The nonlinear terms $F(u) = u^2 u_x + u_{xxx}$ can be decomposed into one infinite series of polynomials given by

$$F(u) = u^2 u_x + u_{xxx} = \sum_{k=0}^{\infty} A_k, \tag{8}$$

where $A_k$ are the so-called Adomian polynomials that can be calculated for all forms of nonlinearity according to algorithms set by Adomian [8-10]. Adomian polynomials are defined by

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[F\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \quad k = 0, 1, 2, \ldots \tag{9}$$

Substituting (7) and (8) into (6) gives rise to

$$\sum_{k=0}^{\infty} u_k(x,t) = f(x) - L^{-1}\left(\sum_{k=0}^{\infty} A_k\right). \tag{10}$$

To determine the components $u_k(x,t)$, $k \geq 0$, we employ the recursive relation

$$\left.\begin{aligned} &u_0(x,t) = f(x), \\ &u_{k+1}(x,t) = -L^{-1}(A_k), \quad k \geq 0, \end{aligned}\right\} \tag{11}$$

where $A_k$ are the Adomian polynomials that represent the nonlinear terms $u^2 u_x + u_{xxx}$ and can be derived by

$$\left.\begin{aligned} &A_0 = u_0^2(u_0)_x + (u_0)_{xxx}, \\ &A_1 = 2u_0(u_0)_x u_1 + u_0^2(u_1)_x + (u_1)_{xxx}, \\ &A_2 = u_1^2(u_0)_x + 2u_0(u_1)_x u_1 + 2u_0(u_0)_x u_2 + u_0^2(u_2)_x + (u_2)_{xxx}, \\ &\cdots \end{aligned}\right\} \tag{12}$$

In view of (11) and (12), we know that all of the components $u_k(x,t)$ can be calculated, and the series solution of $u(x,t)$ follows immediately. The series solution may provide the solution in a closed form if an exact solution exists.

3. **Exact Solutions of the Modified KdV Equation.** We consider the modified KdV equation with the initial condition:

$$u_t + u^2 u_x + u_{xxx} = 0, \quad u(x,0) = \sqrt{6}k \ \mathrm{sech}(kx) \tag{13}$$

where $k$ is an arbitrary constant. Applying the integral operator $L^{-1}$ to both sides of (13) yields

$$u(x,t) = \sqrt{6}k \ \mathrm{sech}(kx) - L^{-1}\left(u^2 u_x + u_{xxx}\right). \tag{14}$$

Substituting the decomposition series (7) for $u(x,t)$ into (14) yields

$$\sum_{k=0}^{\infty} u_k(x,t) = \sqrt{6}k \ \mathrm{sech}(kx) - L^{-1}\left(\sum_{k=0}^{\infty} A_k\right), \tag{15}$$

where $A_k$ are Adomian polynomials that represent the nonlinear terms $u^2u_x + u_{xxx}$. According to the above-mentioned Aomian decomposition method, we have the recursive relation

$$u_0(x,t) = \sqrt{6}k \ \mathrm{sech}(kx), \quad u_{k+1}(x,t) = -L^{-1}A_k, \quad k \geq 0. \tag{16}$$

Substituting (12) into (16) gives

$$\left.\begin{aligned}
u_0(x,t) &= \sqrt{6}k \ \mathrm{sech}(kx) \\
u_1(x,t) &= -L^{-1}A_0 = -L^{-1}\left(-\frac{k^4\sqrt{6}\sinh(kx)}{\cosh^2(kx)}\right) = \frac{k^4\sqrt{6}\sinh(kx)}{\cosh^2(kx)}t \\
u_2(x,t) &= -L^{-1}A_1 = -L^{-1}\left(-\frac{k^7\sqrt{6}t\left(-2+\cosh^2(kx)\right)}{\cosh^3(kx)}\right) \\
&= \frac{k^7\sqrt{6}\left(-2+\cosh^2(kx)\right)}{2\cosh^3(kx)}t^2 \\
u_3(x,t) &= -L^{-1}A_2 = -L^{-1}\left(-\frac{k^{10}\sqrt{6}t^2\sinh(kx)\left(-6+\cosh^2(kx)\right)}{2\cosh^4(kx)}\right) \\
&= \frac{k^{10}\sqrt{6}\sinh(kx)\left(-6+\cosh^2(kx)\right)}{6\cosh^4(kx)}t^3 \\
&\cdots
\end{aligned}\right\} \tag{17}$$

Thus, this gives the solution of (13) in a series form

$$\left.\begin{aligned}
u(x,t) = \sqrt{6}k \ \mathrm{sech}(kx) &+ \frac{k^4\sqrt{6}\sinh(kx)}{\cosh^2(kx)}t + \frac{k^7\sqrt{6}\left(-2+\cosh^2(kx)\right)}{2\cosh^3(kx)}t^2 \\
&+ \frac{k^{10}\sqrt{6}\sinh(kx)\left(-6+\cosh^2(kx)\right)}{6\cosh^4(kx)}t^3 + \cdots
\end{aligned}\right\} \tag{18}$$

Using Taylor series into (18), we obtain the closed form solution

$$u(x,t) = \sqrt{6}k \ \mathrm{sech}\left(k\left(x - k^2 t\right)\right). \tag{19}$$

It is an exact solution for the MKdV Equation (1).

In additional, we can get more exact solutions by adding a constant to the argument in (19). In other words, we introduce the exact solutions

$$u(x,t) = \sqrt{6}k \ \mathrm{sech}\left(k\left(x - k^2 t\right) + c\right) \tag{20}$$

where $c$ is a constant.

These closed form solutions are single-soliton solutions. In order to show that how the Aomian decomposition method is computationally efficient in the course of obtaining single-soliton solutions for Equation (13), we will give detailed numerical solitary solution and the corresponding graphs of solitary in the following section.

4. **Numerical Results for the Modified KdV Equation.** For numerical comparisons, we consider the approximate solution for modified KdV Equation (13). Based on the Adomian decomposition method, we evaluate the approximate solution using the $n$-term approximation, i.e.,

$$Z_n = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots + u_{n-1}(x,t). \tag{21}$$

Table 1 shows the exact solution, approximate solution and the absolute error between them for $k = 2$ and $n = 5$. We display the approximate numerical solution $Z_5$ of Equation (13) in Figure 1, as well as the corresponding numerically exact solution $u(x,t)$ in Figure 2 when $k = 2$.

TABLE 1. The exact solution, approximate solution, absolute error as $k = 2$

| $x_i$ | $t_i$ | Exact solution $u(x,t)$ | Approximate solution $Z_5$ | Absolute error $\mid u(x,t) - Z_5 \mid$ |
|---|---|---|---|---|
| 5 | 0.3 | 0.004903401383831 | 0.004903201303751 | 0.200080079999602e-06 |
| 5 | 0.2 | 0.002203240704845 | 0.002203131704946 | 0.108999898999936e-06 |
| 5 | 0.1 | 9.899799035609665e-04 | 9.899799036806766e-04 | −0.000000119710014e-06 |
| 4 | 0.3 | 0.036231021544848 | 0.036231030564749 | −0.090199009997183e-07 |
| 4 | 0.2 | 0.016279825045851 | 0.016279755037653 | 0.700081980026246e-07 |
| 4 | 0.1 | 0.007315013041587 | 0.007315025143789 | −0.121022020000902e-07 |
| 3 | 0.3 | 0.267516987091128 | 0.267516698113136 | 0.288977991980133e-06 |
| 3 | 0.2 | 0.120274743263007 | 0.120274694283238 | 0.048979768990653e-06 |
| 3 | 0.1 | 0.054049426997956 | 0.054049506878748 | −0.079880792001885e-06 |
| 2 | 0.3 | 1.900697193042293 | 1.900697204236156 | −0.011193863169723e-06 |
| 2 | 0.2 | 0.881595476503612 | 0.881595500914532 | −0.024410919952089e-06 |
| 2 | 0.1 | 0.398723899557676 | 0.398723653868795 | 0.245688881017614e-06 |
| 1 | 0.3 | 4.531592530882302 | 4.531592469031895 | 0.618504074623161e-07 |
| 1 | 0.2 | 4.531592530882302 | 4.531592481763298 | 0.491190039753064e-07 |
| 1 | 0.1 | 2.705638539971261 | 2.705638547800867 | −0.078296058525495e-07 |

Numerical approximations show a high degree of accuracy for approximate solution $Z_n$ for low values of $n$. The solution for modified KdV Equation (13) converges very rapidly by using the Adomian decomposition method. The numerical results also demonstrate the advantage of this method that has accurate approximation with few terms. Last but not least, Adomian decomposition method is not affected by computational round-off errors, and it is not confronted with the necessity for larger computer memory and time.

5. **Conclusions.** In summary, we have presented a scheme to obtain solutions of the MKdV equation by using the known Adomian decomposition method. Adomian decomposition method was employed successfully to develop exact solution for the modified KdV equation. The computations associated with the explicit exact solitary solutions discussed above were performed by using symbolic computation system, Maple. Perhaps other solitary solutions of the MKdV equation may be constructed by using the decomposition method.

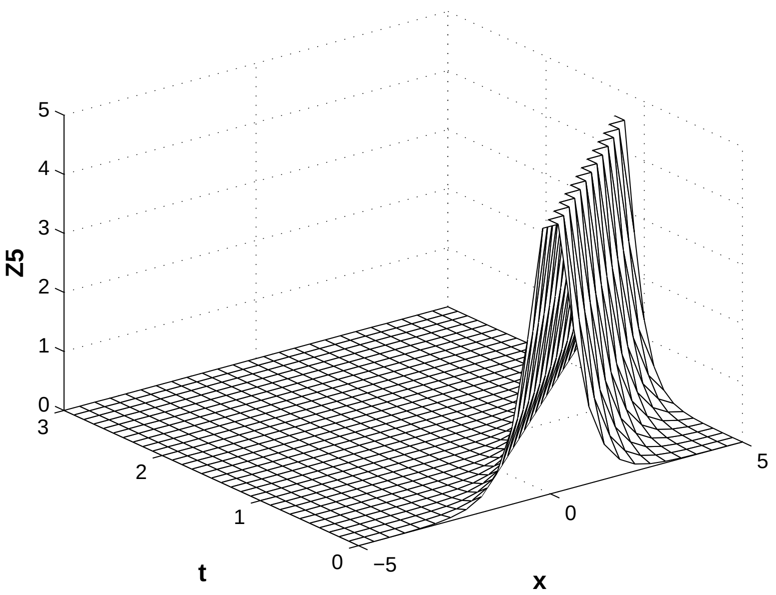


FIGURE 1. Approximate solution

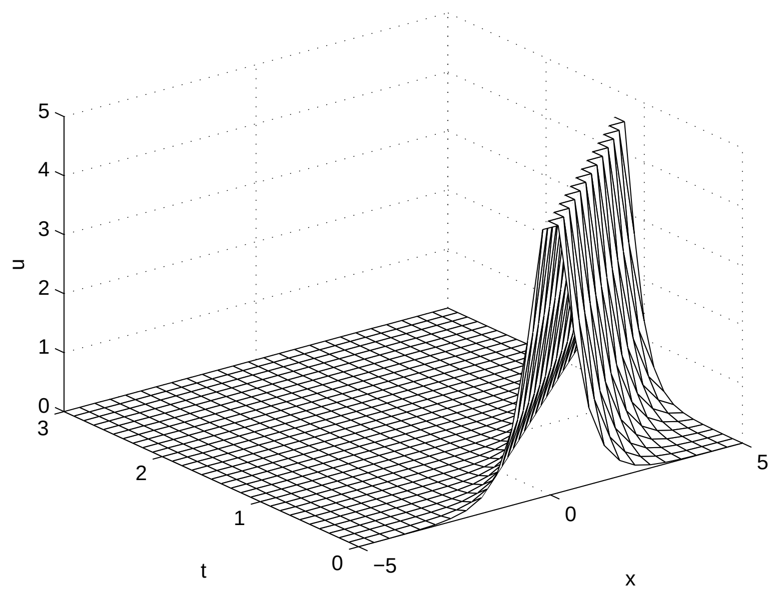


FIGURE 2. Exact solution

**Acknowledgment.** This work was supported by the National Natural Science Foundation of China under Grant (No. 11571325, No. 11271126). The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

**REFERENCES**

[1] R. Hirota, Exact solutions of the KdV equation for multiple collisions of solitons, *Physical Review Letters*, vol.27, no.18, pp.1192-1193, 1971.
[2] C. Yan, A simple transformation for nonlinear waves, *Physics Letters A*, vol.24, nos.1-2, pp.77-84, 1996.

[3] W. Malfliet, Solitary wave solutions of nonlinear wave equations, *American Journal of Physics*, vol.60, no.7, pp.650-654, 1992.

[4] M. Wang, Exact solutions for a compound KdV-Burgers equation, *Physics Letters A*, vol.213, nos.5-6, pp.279-287, 1996.

[5] S. A. El-Wakil and M. A. Abdou, New exact traveling wave solutions of two nonlinear physical models, *Nonlinear Analysis: Theory, Methods & Applications*, vol.68, no.2, pp.235-245, 2008.

[6] E. Shek and K. Chow, The discrete modified Korteweg-de-Vries equation with non-vanishing boundary conditions: Interactions of solitons, *Chaos, Solitons & Fractals*, vol.36, no.2, pp.296-302, 2008.

[7] M. Miurs, *Backlun Transformation*, Springer, Berlin, 1978.

[8] G. Adomian, A review of the decomposition method in applied mathematics, in *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston, 1994.

[9] G. Adomian, A review of the decomposition method in applied mathematics, *Journal of Mathematical Analysis and Applications*, vol.135, no.2, pp.501-544, 1998.

[10] G. Adomian, *Nonlinear Stochastic Operator Equations*, Academic Press, San Diego, 1986.

[11] A. M. Wazwaz, Exact special solutions with solitary patterns for the nonlinear dispersive $K(m,n)$ equations, *Chaos, Solitons & Fractals*, vol.13, no.1, pp.161-170, 2002.

[12] A. M. Wazwaz, New solitary-wave special solutions with compact support for the nonlinear dispersive $K(m,n)$ equations, *Chaos, Solitons & Fractals*, vol.13, no.2, pp.321-330, 2002.

[13] M. A. Abdou, Solitary solutions for nonlinear differential-difference equations via Adomain decomposition method, *International Journal of Nonlinear Science*, vol.12, no.1, pp.29-35, 2011.

[14] M. Sheikholeslami, D. D. Ganji, H. R. Ashorynejad and H. R. Rokni, Analytical investigation of Jeffery-Hamel flow with high magnetic field and nanoparticle by Adomian decomposition method, *Applied Mathematics and Mechanics*, vol.33, no.1, pp.25-36, 2012.

[15] E. Babolian, A. R. Vahidi and A. Shoja, An efficient method for nonlinear fractional differential equations: Combination of the Adomian decomposition method and spectral, *Indian Journal of Pure and Applied Mathematics*, vol.45, no.6, pp.1017-1028, 2014.

[16] B. S. Alkahtani, P. Goswami and O. J. Algahtanic, Adomian decomposition method for $n$-dimensional diffusion model in fractal heat transfer, *Journal of Nonlinear Science and Applications*, vol.9, no.5, pp.2982-2985, 2016.

[17] S. Paul, S. P. Mondal and P. Bhattacharya, Numerical solution of Lota Volterra prey predator model by using Runge-Kutta-Fehlberg method and Laplace Adomian decomposition method, *Alexandria Engineering Journal*, vol.55, no.2, pp.613-617, 2016.

[18] H. Eltayeb, A. Kilicman and S. Mesloub, Application of the double Laplace Adomian decomposition method for solving linear singular one dimensional thermo-elasticity coulped system, *Journal of Nonlinear Science and Applications*, vol.10, no.1, pp.278-289, 2017.