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STATISTICAL INFERENCE OF THE MEASURE OF PERFORMANCE FOR GENERALIZED EXPONENTIAL PRODUCTS UNDER PROGRESSIVE TYPE II RIGHT CENSORING SCHEME

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Abstract. *The stress-strength model has substantial interest and usefulness in various areas of engineering, psychology, genetics and clinical trials. The generalized exponential distribution can be used quite effectively to analyze skewed data sets. In addition, comparing the performance of products in the quality testing experiment, progressive type II right censoring scheme is quite useful in many practical situations where budget constraints are in place or there is a demand for rapid testing. This study constructs a maximum likelihood estimator (MLE) of $R = P[X > Y]$ when the quality characters of two products X and Y are independent generalized exponential distribution with the progressive type II right censored sample. The MLE of $R = P[X > Y]$ is then utilized to develop the new hypothesis testing procedure. Moreover, the new testing procedure can be employed by managers to assess whether X is superior to Y in quality performance. Finally, one numerical example is utilized to illustrate the use of the testing procedure.*
Keywords: Progressive censored sample, Generalized exponential distribution, Maximum likelihood estimator

1. **Introduction.** The stress-strength model has substantial interest and usefulness in various areas of engineering, psychology, genetics and clinical trials (see [1]). This model involves two independent random variables X and Y, and the probability $R = P[X > Y]$, for example, (a) in mechanical reliability of a system, if X is the strength of a system which is subject to stress Y, then R is a measure of performance for the system. The system fails, if the applied stress is greater than its strength at anytime; (b) if Y represents a patient's survival time when he is treated with drug A and X represents another patient's survival time when he is treated with drug B, then drug B could be preferred to drug A if $R = P[X > Y] > 0.5$ (see [2]); (c) if X and Y represent lifetimes of two devices, then R is the probability that the lifetime X is greater than the lifetime Y (see [3]). The problem of estimating of $R = P[X > Y]$ has been widely used in much related statistical literatures, for example, exponential distribution (see [3,4]), Weibull distribution (see [2,5-7]), and generalized exponential distribution (see [8-10]). The theoretical and practical results on the theory and applications of the stress-strength relationships in industrial and economic systems during the last decades are collected and digested in [1].

The generalized exponential distribution was introduced by [11,12]. The generalized exponential distribution with the shape parameter θ and the scale parameter λ will be denoted by $GE(\theta, \lambda)$ and the corresponding probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) are as follows respectively:

$$f(x) = \frac{\theta}{\lambda}\left[1 - \exp\left(-\frac{x}{\lambda}\right)\right]^{\theta-1} \exp\left(-\frac{x}{\lambda}\right), \quad x > 0,\ \theta > 0,\ \lambda > 0, \tag{1}$$

and

$$F(x) = \left[1 - \exp\left(-\frac{x}{\lambda}\right)\right]^{\theta}, \quad x > 0,\ \theta > 0,\ \lambda > 0, \tag{2}$$

where θ is the shape parameter and λ is the scale parameter. For $\theta = 1$, the generalized exponential distribution reduces to the exponential distribution with mean λ. For $\theta \leq 1$, the probability density function is a strictly decreasing function and for $\theta > 1$, it has a unimodal shape. These probability density functions are illustrated in [12]. It is clear that the probability density functions of generalized exponential distribution are always right skewed and it is observed that the generalized exponential distribution can be used quite effectively to analyze skewed data sets (see [11,13]).

In this paper, we consider the case of the progressive type II right censoring. Progressive type II right censoring is a useful scheme in which a specific fraction of individuals at risk may be removed from the experiment at each of several ordered failure times (see [14]). The experimenter can remove units from a life test at various stages during the experiments, possibly resulting in a saving of costs and times (see [15]). The familiar complete and type II right censored samples are special cases of this scheme. A schematic illustration is depicted as follows, where $x_{1,n}, x_{2,n}, \ldots, x_{m,n}$ denote the observed failure times and $R_1, R_2, \ldots, R_m$ denote the corresponding numbers of units removed (withdrawn) from the test. Let m be the number of failures observed before termination and $x_{1,n} \leq x_{2,n} \leq \cdots \leq x_{m,n}$ be the observed ordered lifetimes. Let R_i denote the number of units removed at the time of the ith failure, $0 \leq R_i \leq n - \sum_{j=1}^{i-1} R_j - i$, $i = 2, 3, \ldots, m-1$, with $0 \leq R_1 \leq n-1$ and $R_m = n - \sum_{j=1}^{m-1} R_j - m$, where R_i's and m are prespecified integers (see [16]).

To utilize the probability $R = P[X > Y]$ in assessing the quality performance of products more generally and accurately. This study constructs a maximum likelihood estimator (*MLE*) of $R = P[X > Y]$ when the quality characters of two products X and Y are independent generalized exponential distribution with the progressive type II right censored sample. The *MLE* of $R = P[X > Y]$ is then utilized to develop the new hypothesis testing procedure. Moreover, the new testing procedure can be employed by managers to assess whether X is superior to Y in quality performance.

The rest of this paper is organized as follows. Section 2 presents the *MLE* of $R = P[X > Y]$ under $X \sim GE(\theta_1, \lambda)$ and $Y \sim GE(\theta_2, \lambda)$, respectively. Section 3 develops a confidence interval of $R = P[X > Y]$ and a new hypothesis testing procedure. Finally, one numerical example and concluding remarks are made in Section 4 and Section 5, respectively.

2. The Maximum Likelihood Estimator of *R*.

Suppose that the quality characters of two products X and Y follow $GE(\theta_1, \lambda)$ and $GE(\theta_2, \lambda)$, respectively, where X and Y are independent random variables. Therefore, by using (1), we obtain

$$R = P[X > Y] = \frac{\theta_1}{\theta_1 + \theta_2}. \tag{3}$$

Next, let $X_{1,n_1} \leq X_{2,n_1} \leq \cdots \leq X_{m_1,n_1}$ be the corresponding progressive type II right censored sample from $GE(\theta_1, \lambda)$ with progressive censoring scheme $R^* = (R_1, R_2, \ldots, R_{m_1})$ and $Y_{1,n_2} \leq Y_{2,n_2} \leq \cdots \leq Y_{m_2,n_2}$ be the corresponding progressive type II right censored sample from $GE(\theta_2, \lambda)$ with progressive censoring scheme $R' = (R'_1, R'_2, \cdots, R'_{m_2})$. Then the log-likelihood function of the progressive type II right censored sample is given by

$$\begin{aligned}
\ln L(\Lambda) = {} & \ln C_1 + \ln C_2 + m_1 \ln \theta_1 + m_2 \ln \theta_2 - (m_1 + m_2) \ln \lambda \\
& + (\theta_1 - 1) \sum_{i=1}^{m_1} \ln \left(1 - \exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right) - \sum_{i=1}^{m_1} \frac{x_{i,n_1}}{\lambda} \\
& + \sum_{i=1}^{m_1} R_i \ln \left[1 - \left(1 - \exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right)^{\theta_1}\right] \\
& + (\theta_2 - 1) \sum_{j=1}^{m_2} \ln \left(1 - \exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right) - \sum_{j=1}^{m_2} \frac{y_{j,n_2}}{\lambda} \\
& + \sum_{j=1}^{m_2} R'_i \ln \left[1 - \left(1 - \exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right)^{\theta_2}\right],
\end{aligned} \tag{4}$$

where $\Lambda = (\theta_1, \theta_2, \lambda)$, $C_1 = n_1(n_1 - R_1 - 1) \cdots (n_1 - R_1 - R_2 - \cdots - R_{m_1-1} - m_1 + 1)$, $C_2 = n_2(n_2 - R'_1 - 1) \cdots (n_2 - R'_1 - R'_2 - \cdots - R'_{m_2-1} - m_2 + 1)$, $x_{1,n_1} \le x_{2,n_1} \le \cdots \le x_{m_1,n_1}$, $y_{1,n_2} \le y_{2,n_2} \le \cdots \le y_{m_2,n_2}$. Since θ_1, θ_2 and λ are unknown, by solving the equations $\frac{\partial \ln L(\Lambda)}{\partial \theta_1} = 0$, $\frac{\partial \ln L(\Lambda)}{\partial \theta_2} = 0$ and $\frac{\partial \ln L(\Lambda)}{\partial \lambda} = 0$, we obtain that the *MLE*s $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\lambda}$ satisfy the following nonlinear Equations (5), (6) and (7) as given by

$$\begin{aligned}
& \frac{m_1}{\hat{\theta}_1} + \sum_{i=1}^{m_1} \ln \left(1 - \exp\left(-\frac{x_{i,n_1}}{\hat{\lambda}}\right)\right) \\
& - \sum_{i=1}^{m_1} \frac{R_i \left(1 - \exp\left(-\frac{x_{i,n_1}}{\hat{\lambda}}\right)\right)^{\hat{\theta}_1} \ln \left(1 - \exp\left(-\frac{x_{i,n_1}}{\hat{\lambda}}\right)\right)}{1 - \left(1 - \exp\left(-\frac{x_{i,n_1}}{\hat{\lambda}}\right)\right)^{\hat{\theta}_1}} = 0,
\end{aligned} \tag{5}$$

$$\begin{aligned}
& \frac{m_2}{\hat{\theta}_2} + \sum_{j=1}^{m_2} \ln \left(1 - \exp\left(-\frac{y_{j,n_2}}{\hat{\lambda}}\right)\right) \\
& - \sum_{j=1}^{m_2} \frac{R'_j \left(1 - \exp\left(-\frac{y_{j,n_2}}{\hat{\lambda}}\right)\right)^{\hat{\theta}_2} \ln \left(1 - \exp\left(-\frac{y_{j,n_2}}{\hat{\lambda}}\right)\right)}{1 - \left(1 - \exp\left(-\frac{y_{j,n_2}}{\hat{\lambda}}\right)\right)^{\hat{\theta}_2}} = 0,
\end{aligned} \tag{6}$$

$$\begin{aligned}
& - \frac{m_1 + m_2}{\hat{\lambda}} - \left(\hat{\theta}_1 - 1\right) \sum_{i=1}^{m_1} \frac{\frac{x_{i,n_1}}{\hat{\lambda}^2} \exp\left(-\frac{x_{i,n_1}}{\hat{\lambda}}\right)}{1 - \exp\left(-\frac{x_{i,n_1}}{\hat{\lambda}}\right)} \\
& + \frac{\sum_{i=1}^{m_1} x_{i,n_1}}{\hat{\lambda}^2} + \sum_{i=1}^{m_1} \frac{R_i \hat{\theta}_1 \left(1 - \exp\left(-\frac{x_{i,n_1}}{\hat{\lambda}}\right)\right)^{\theta_1 - 1} \frac{x_{i,n_1} \exp\left(-\frac{x_{i,n_1}}{\hat{\lambda}}\right)}{\hat{\lambda}^2}}{1 - \left(1 - \exp\left(-\frac{x_{i,n_1}}{\hat{\lambda}}\right)\right)^{\hat{\theta}_1}} \\
& - \left(\hat{\theta}_2 - 1\right) \sum_{j=1}^{m_2} \frac{\frac{y_{j,n_2}}{\hat{\lambda}^2} \exp\left(-\frac{y_{j,n_2}}{\hat{\lambda}}\right)}{1 - \exp\left(-\frac{y_{j,n_2}}{\hat{\lambda}}\right)} + \frac{\sum_{j=1}^{m_2} y_{j,n_2}}{\hat{\lambda}^2} \\
& + \sum_{j=1}^{m_2} \frac{R'_j \hat{\theta}_2 \left(1 - \exp\left(-\frac{y_{j,n_2}}{\hat{\lambda}}\right)\right)^{\hat{\theta}_2 - 1} \frac{y_{j,n_2} \exp\left(-\frac{y_{j,n_2}}{\hat{\lambda}}\right)}{\hat{\lambda}^2}}{1 - \left(1 - \exp\left(-\frac{y_{j,n_2}}{\hat{\lambda}}\right)\right)^{\hat{\theta}_2}} = 0.
\end{aligned} \tag{7}$$

By using the invariance of *MLE* (see [17]), the *MLE* $\hat{R}$ of R can be written as

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}, \tag{8}$$

where the *MLE*s $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\lambda}$ can also be found by gradient descent numerical method to solve the nonlinear Equations (5), (6) and (7) (see [18]).

The asymptotic normal distribution for the $\hat{R}$ can be obtained in large sample theory. From the log-likelihood function in (4), we have

$$\frac{\partial^2 \ln L(\Lambda)}{\partial \theta_1^2} = -\frac{m_1}{\theta_1^2} - \sum_{i=1}^{m_1} \frac{R_i \left(1-\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right)^{\theta_1} \ln^2\left(1-\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right)}{\left[1-\left(1-\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right)^{\theta_1}\right]^2}, \tag{9}$$

$$\frac{\partial^2 \ln L(\Lambda)}{\partial \theta_2^2} = -\frac{m_2}{\theta_2^2} - \sum_{j=1}^{m_2} \frac{R'_j \left(1-\exp\left(\frac{-y_{j,n_2}}{\lambda}\right)\right)^{\theta_2} \ln^2\left(1-\exp\left(\frac{-y_{j,n_2}}{\lambda}\right)\right)}{\left[1-\left(1-\exp\left(\frac{-y_{j,n_2}}{\lambda}\right)\right)^{\theta_2}\right]^2}, \tag{10}$$

$$\begin{aligned}
&\frac{\partial^2 \ln L(\Lambda)}{\partial \lambda^2} \\
&= \frac{m_1}{\lambda^2} - (\theta_1 - 1)\sum_{i=1}^{m_1} \frac{\frac{x_{i,n_1}}{\lambda^3}\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\left(-2+\frac{x_{i,n_1}}{\lambda}+2\exp\left(\frac{-x_{i,n_1}}{\lambda}\right)\right)}{\left(1-\exp\left(\frac{-x_{i,n_1}}{\lambda}\right)\right)^2} - \frac{2\sum_{i=1}^{m_1} x_{i,n_1}}{\lambda^3} \\
&+ \sum_{i=1}^{m_1} \frac{R_i\theta_1\left(1-\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right)^{\theta_1-2}\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\frac{x_{i,n_1}}{\lambda^3}}{\left[1-\left(1-\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right)^{\theta_1}\right]^2}\Bigg\{-\theta_1\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\frac{x_{i,n_1}}{\lambda} \\
&+ \left[1-\left(1-\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right)^{\theta_1}\right]\cdot\left[\frac{x_{i,n_1}}{\lambda} - 2 + 2\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right]\Bigg\} \qquad (11)\\
&+ \frac{m_2}{\lambda^2} - (\theta_2 - 1)\sum_{j=1}^{m_2} \frac{\frac{y_{j,n_2}}{\lambda^3}\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\left(-2+\frac{y_{j,n_2}}{\lambda}+2\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right)}{\left(1-\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right)^2} - \frac{2\sum_{j=1}^{m_2} y_{j,n_2}}{\lambda^3} \\
&+ \sum_{j=1}^{m_2} \frac{R'_i\theta_2\left(1-\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right)^{\theta_2-2}\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\frac{y_{j,n_2}}{\lambda^3}}{\left[1-\left(1-\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right)^{\theta_2}\right]^2}\Bigg\{-\theta_2\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\frac{y_{j,n_2}}{\lambda} \\
&+ \left[1-\left(1-\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right)^{\theta_2}\right]\left[\frac{y_{j,n_2}}{\lambda} - 2 + 2\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right]\Bigg\},
\end{aligned}$$

$$\frac{\partial^2 \ln L(\Lambda)}{\partial \theta_2 \partial \theta_1} = 0, \tag{12}$$

$$\begin{aligned}
\frac{\partial^2 \ln L(\Lambda)}{\partial \lambda \partial \theta_1} &= -\sum_{i=1}^{m_1} \frac{\frac{x_{i,n_1}}{\lambda^2}\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)}{1-\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)} - \sum_{j=1}^{m_1}\Bigg\{R_i\left(1-\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right)^{\theta_1-1} \\
&\cdot \exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\left(\frac{x_{i,n_1}}{\lambda^2}\right)\cdot\Bigg[-\theta_1\ln\left(1-\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right) - 1 \qquad (13)\\
&+ \left(1-\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right)^{\theta_1}\Bigg] \Bigg/ \left[1-\left(1-\exp\left(-\frac{x_{i,n_1}}{\lambda}\right)\right)^{\theta_1}\right]^2\Bigg\},
\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ln L(\Lambda)}{\partial\lambda\partial\theta_2} = &-\sum_{j=1}^{m_2}\frac{\frac{y_{j,n_2}}{\lambda^2}\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)}{1-\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)} - \sum_{j=1}^{m_2}\left\{R_j'\left(1-\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right)^{\theta_2-1}\right.\\ &\cdot\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\left(\frac{y_{j,n_2}}{\lambda^2}\right)\cdot\left[-\theta_2\ln\left(1-\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right)-1\right.\\ &\left.\left.+\left(1-\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right)^{\theta_2}\right]\Big/\left[1-\left(1-\exp\left(-\frac{y_{j,n_2}}{\lambda}\right)\right)^{\theta_2}\right]^2\right\}.\end{aligned} \quad (14)$$

By using [19], the Fisher information matrix is given by

$$I(\Lambda) = [I_{ij}(\Lambda)]_{3\times 3}, \quad (15)$$

where $I_{11}(\Lambda) = E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial\theta_1^2}\right)$, $I_{12}(\Lambda) = I_{21}(\Lambda) = E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial\theta_1\partial\theta_2}\right)$, $I_{13}(\Lambda) = I_{31}(\Lambda) = E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial\theta_1\partial\lambda}\right)$, $I_{22}(\Lambda) = E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial\theta_2^2}\right)$, $I_{23}(\Lambda) = I_{32}(\Lambda) = E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial\theta_2\partial\lambda}\right)$, $I_{33}(\Lambda) = E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial\lambda^2}\right)$. Under some regularity conditions (see [20]), $\hat{\Lambda}$ is asymptotically trivariate normal distribution with mean Λ and covariance matrix $I^{-1}(\Lambda)$, i.e., $\hat{\Lambda}\xrightarrow{D}N(\Lambda, I^{-1}(\Lambda))$. Let $R = \frac{\theta_1}{\theta_1+\theta_2} = g(\theta_1,\theta_2,\lambda) = g(\Lambda)$, by the delta method (see [21]), we have

$$\hat{R}\xrightarrow{D}N\left(R, Var(\hat{R})\right), \quad (16)$$

where $Var\left(\hat{R}\right) = JI^{-1}(\Lambda)J'$, $J = \left[\frac{\partial g(\Lambda)}{\partial\theta_1}, \frac{\partial g(\Lambda)}{\partial\theta_2}, \frac{\partial g(\Lambda)}{\partial\lambda}\right]$, $\frac{\partial g(\Lambda)}{\partial\theta_1} = \frac{\theta_2}{(\theta_1+\theta_2)^2}$, $\frac{\partial g(\Lambda)}{\partial\theta_2} = -\frac{\theta_1}{(\theta_1+\theta_2)^2}$, $\frac{\partial g(\Lambda)}{\partial\lambda} = 0$ and $I(\Lambda)$ as the above definition.

3. **The Confidence Interval of *R*.** Owing to the sampling error, the point estimate of the measure of performance R cannot be employed directly to determine whether X is superior to Y in quality performance. Thus, a confidence interval is needed to objectively assess whether X is superior to Y in quality performance. Given the specified significance level α^*, the level $100(1-\alpha^*)\%$ confidence interval and one-sided confidence interval for R can be derived as follows.

With the pivotal quantity $\frac{\hat{R}-R}{\sqrt{Var(\hat{R})}}$ and by using the asympotic result of [22, p.549], so we can obtain

$$\frac{\hat{R}-R}{\sqrt{\widehat{Var}(\hat{R})}}\xrightarrow{D}N(0,1), \quad (17)$$

where $\widehat{Var}(\hat{R}) = J\,I_0^{-1}(\Lambda)J'\big|_{\Lambda=\hat{\Lambda}}$, $J = \left[\frac{\partial g(\Lambda)}{\partial\theta_1}, \frac{\partial g(\Lambda)}{\partial\theta_2}, \frac{\partial g(\Lambda)}{\partial\lambda}\right]$ and $I_0(\Lambda) = \left[I_{ij}^*(\Lambda)\right]_{3\times 3}$, $I_{11}^*(\Lambda) = -\frac{\partial^2 \ln L(\Lambda)}{\partial\theta_1^2}$, $I_{12}^*(\Lambda) = I_{21}^*(\Lambda) = -\frac{\partial^2 \ln L(\Lambda)}{\partial\theta_1\partial\theta_2}$, $I_{13}^*(\Lambda) = I_{31}^*(\Lambda) = -\frac{\partial^2 \ln L(\Lambda)}{\partial\theta_1\partial\lambda}$, $I_{22}^*(\Lambda) = \left(-\frac{\partial^2 \ln L(\Lambda)}{\partial\theta_2^2}\right)$, $I_{23}^*(\Lambda) = I_{32}^*(\Lambda) = -\frac{\partial^2 \ln L(\Lambda)}{\partial\theta_2\partial\lambda}$, $I_{33}^*(\Lambda) = -\frac{\partial^2 \ln L(\Lambda)}{\partial\lambda^2}$.

Hence, we can know the level $100\,(1-\alpha^*)\%$ confidence interval for R as given by

$$\left(\hat{R} - Z_{\alpha^*/2}\sqrt{\widehat{Var}\left(\hat{R}\right)}, \hat{R} + Z_{\alpha^*/2}\sqrt{\widehat{Var}\left(\hat{R}\right)}\right), \quad (18)$$

where Z_{α^*} represents the lower $100\,(1-\alpha^*)\%$th percentile of standard normal distribution. We can then employ the level $100\,(1-\alpha^*)\%$ confidence interval to test whether $R \neq R_0$ in large sample. The hypothesis of the proposed testing procedure about R can be organized as the null hypothesis $H_0: R = R_0$ and the alternative hypothesis $H_1: R \neq R_0$, where in general, given $R_0 = 0.5$. At the level of significance α^*, the decision rule of statistical test is "If $R_0 \notin \left(\hat{R} - Z_{\alpha^*/2}\sqrt{\widehat{Var}\left(\hat{R}\right)}, \hat{R} + Z_{\alpha^*/2}\sqrt{\widehat{Var}\left(\hat{R}\right)}\right)$, it is concluded that $R \neq R_0$". In addition, we can also construct the level $100\,(1-\alpha^*)\%$ one-sided confidence interval to test whether $R > R_0$ in large sample. The hypothesis of the proposed testing procedure about R can be organized as the null hypothesis $H_0 : R \leq R_0$ and the alternative hypothesis $H_1 : R > R_0$, where in general, given $R_0 = 0.5$. At the α^* level of significance, the decision rule of statistical test is "If $R_0 \notin \left(\hat{R} - Z_{\alpha^*}\sqrt{\widehat{Var}(\hat{R})}, 1\right)$, it is concluded that $R > R_0$".

4. **A Numerical Example.** In this section, we propose the new hypothesis testing procedure to a simulation data. One numerical example illustrates the use of the new hypothesis testing procedure.

Example (*Simulated data*). We discuss $X \sim GE(\theta_1, \lambda)$ and $Y \sim GE(\theta_2, \lambda)$ with the unknown scale parameter λ case and the unknown shape parameters θ_1 and θ_2 case. We assume that the scale parameter λ of two independent X and Y are identical and unknown λ, where X (Data Set 1) is simulated data from the generalized exponential distribution $GE(3, 4)$ with progressive censored schemes $R^* = (0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 2, 2, 0, 2, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, $m_1 = 30$, $n_1 = 50$ and Y (Data Set 2) is simulated data from the generalized exponential distribution $GE(2, 4)$ with progressive censored schemes $R' = (0, 0, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2)$, $m_2 = 30$, $n_2 = 40$.

X (Data Set 1) = {1.25113, 1.38738, 2.81931, 2.94886, 2.95702, 3.01037, 3.18129, 3.24973, 3.30575, 3.52128, 3.64958, 3.67013, 3.75384, 3.98662, 4.52966, 4.65302, 5.02475, 5.09347, 5.22836, 5.51711, 5.58028, 5.69626, 6.76844, 6.83901, 7.74378, 7.80644, 8.94753, 9.99827, 10.8974, 12.0734}.

Y (Data Set 2) = {0.31815, 0.80650, 0.91335, 0.99546, 1.38503, 1.53092, 1.89336, 2.26590, 3.16380, 3.20775, 3.24478, 3.57478, 3.71605, 3.82064, 3.99841, 4.06225, 4.13572, 4.28308, 4.75622, 5.36246, 5.56364, 5.97654, 6.03524, 6.09644, 6.32965, 9.12161, 9.44744, 9.96717, 15.4553, 15.8147}.

By numerical method to solve the nonlinear Equations (5), (6) and (7), we can attain the maximum likelihood estimates $\hat{\theta}_1 = 3.06943$, $\hat{\theta}_2 = 1.83609$ and $\hat{\lambda} = 4.17272$. Moreover, by (8), we have $\hat{R} = \frac{3.06943}{3.06943+1.83609} = 0.62571$. By (18), the level 95% confidence interval for R is given by

$$\left(\hat{R} - Z_{0.025}\sqrt{\widehat{Var}(\hat{R})}, \hat{R} + Z_{0.025}\sqrt{\widehat{Var}(\hat{R})}\right) = (0.524345, 0.727075)$$

where $Z_{0.025} = 1.96$ and $\widehat{Var}\left(\hat{R}\right) = 0.002674643$.

In addition, we also can construct the level 95% one-sided confidence interval for R is given by

$$\left(\hat{R} - Z_{0.05}\sqrt{\widehat{Var}(\hat{R})}, 1\right) = (0.540636, 1)$$

where $Z_{0.05} = 1.645$ and $\widehat{Var}(\hat{R}) = 0.002674643$.

The hypothesis of the proposed testing procedure about R can be organized as the null hypothesis $H_0 : R \leq 0.5$ and the alternative hypothesis $H_1 : R > 0.5$. Because of $0.5 \notin (0.540636, 1)$, we can reject $H_0 : R \leq 0.5$, and it is concluded that $R = P[X > Y] > 0.5$. That is, at the 0.05 level of significance, X is superior to Y in tension.

5. **Conclusions.** In this study, we construct a maximum likelihood estimator (*MLE*) of $R = P[X > Y]$ when the quality characters of two products X and Y are independent generalized exponential distribution with the progressive type II right censored sample. The *MLE* of $R = P[X > Y]$ is then utilized to develop the new hypothesis testing procedure. The new testing procedure can be employed by managers to assess whether X is superior to Y in quality performance. In future research on this problem, it would

be interesting to deal with the exponentiated Weibull products based on the progressive type II right censored sample.

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