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UNCERTAIN MAXIMUM WEIGHT INDEPENDENT SET PROBLEM

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Abstract. *The research of classical maximum weight independent set problem always assumes that the vertex weights are crisp values. However, in the real-life application, the maximum weight independent set problem often comes with uncertainty for lacking of the information about the vertex weights. Based on the uncertainty theory, this paper studies a maximum weight independent set problem under an uncertain environment, in which the vertex weights are assumed to be uncertain variables. First, a belief degree constrained programming model is constructed for the problem. Furthermore, it is proved that there exists an equivalence relationship between uncertain maximum weight independent set model and the corresponding classic deterministic model. We also find that the maximum weight of belief degree constrained programming model is actually decreasing with respect to the predetermined confidence level. Finally, a numerical example is given to show the application of the model.*
Keywords: Independent set problem, Network optimization, Uncertain programming, Uncertainty theory

1. **Introduction.** Independent set problem, known as an important network optimization problem, was employed in many scientific and engineering applications, such as coding theory [4], combinatorial auctions [20], and wireless network [21]. For a graph $G = (V, E)$, a subset I of V is said to be an independent set if no vertices in I are adjacent. The maximum independent set problem (MISP) is the problem of finding an independent set with largest cardinality in a given graph. An extension of the maximum independent set problem is called the maximum weight independent set problem (MWISP), which is to find an independent set with the maximum weight in a given weighted graph. In this paper, we mainly focused on the MWISP.

Garey and Johnson [9] proved the MWISP is the core issue in NP-hard problems. In the following years, some heuristic or metaheuristic solving techniques have been developed for the MWISP. For example, Feo and Resende [5] studied a greedy randomized adaptive search procedure for maximum independent set of a graph. After that, Saha et al. [19] presented an efficient parallel random access machine (PRAM) algorithm to find a maximum weighted independent set of a permutation graph. Nayeem and Pal [17] used the genetic algorithm to find the MWISP of a weighted graph.

It is worth pointing out that all of the above mentioned researches are concerned with the MWISP in a deterministic environment, in which the vertex weights are assumed to be crisp values. However, because of technical or economical reasons, the vertex weights are indeterminacy in many situations. In these cases, it is not suitable to employ classical methods to study the MWISP. Hence, some researchers deemed such indeterminacy behaves like randomness. Based on this assumption, a lot of researches have been presented within the framework of probability theory. For example, Krivelevich et al. [11] investigated the probability of independent sets in random graphs. In addition, Gamarnik et al. [6] studied the maximum weight independent sets in sparse random graphs. Beame et al. [1] discussed the resolution complexity of independent sets and vertex covers in random graphs.

However, the vertex weights of a graph may not be accurately measured according to probability theory in reality. In fact, we may not get probability distribution of vertex weights due to lack of information in some emergency case. In such case, we have to invite some domain experts to give the belief degrees of the vertex weights. According to Nobelist Kahneman and his partner Tversky [10], human beings usually overweigh unlikely events, and thus the belief degree based on experts' estimations may be far from the cumulative frequency. In 2012, Liu [14] pointed out that if we insist on dealing with the belief degree by using probability theory in this situation, some counterintuitive phenomena may occur. Therefore, in this situation we have no choice but to use the uncertainty theory founded by Liu [12] to deal with the belief degree. The interested readers may refer to Gao [7], Gao and Chen [8], Liu [15], Liu et al. [16] about comprehensive development of uncertainty theory.

As an important application of uncertainty theory, uncertain network was pioneered by Liu [13] for modeling project scheduling problem with uncertain duration times. Since Liu's distinguished work, uncertain network has been extensively researched by several scholars. For example, Zhang and Peng [22] proposed an uncertain programming model for Chinese postman problem on uncertain weighted network. Chen et al. [2] investigated the minimum weight vertex covering problem with uncertain vertex weights. Chen et al. [3] studied the bicriteria solid transportation problem under an uncertain environment in which the supplies, demands, conveyance capacities, transportation cost and transportation time were supposed to be uncertain variables. Zhang et al. [23] investigated the fixed charge solid transportation problem under uncertainty. In Liu's work [13], the uncertain network was a network in which the arc capacities or lengthes were uncertain variables. As an extension of uncertain network proposed by Liu [13], Peng et al. [18] gave the definition of uncertain weighted network in 2014, that is, a network has uncertain vertex weights or/and uncertain edge weights. In the work of Peng et al. [18], the uncertain network optimization was defined as the study of network optimization with uncertain data for decision making under the presence of uncertainties.

None of the previous works addressed the maximum weight independent set problem (MWISP) in uncertain environment. In this sense, the first contribution of this paper is that we introduce uncertainty theory to the MWISP. To be more precise, the vertex weights are described by uncertain variables. The second contribution of this paper is that the optimistic value criterion is used to rank the uncertain variables, and accordingly an uncertain programming model is proposed. After presenting a study of the model, we have moved on to provide some related theorems. The third contribution is that we prove that the proposed model can be transformed into its corresponding deterministic form, which implies an approach to solving the model. The relationship between the maximum weight of belief degree constrained programming model and the predetermined confidence level is investigated. We have further carried out a numerical experiment to show the application of the model.

The rest of this paper is organized as follows. For facilitating the understanding of the paper, some basic concepts and results related to uncertainty theory are outlined and the classic maximum weight independent set problem is briefly reviewed in Section 2. Section 3 proposes the concept of α-maximum independent set among uncertain weight independent set. Next we present a belief degree constrained programming model for the problem. For the sake of illustrating the modeling idea of the paper, a numerical example will be presented in Section 4. Finally, Section 5 gives our conclusions.

2. **Preliminary.** In this section, some basic concepts and results of uncertainty theory are recalled, and the classic maximum weight independent set problem is briefly reviewed.

2.1. **Uncertainty theory.** Assume that Γ is a nonempty set, and L is a σ-algebra over Γ. Each element Λ in L is called an event. A set function $M : L \to [0, 1]$ is called an uncertain measure if it satisfies the following three axioms (Liu [12]):
Axiom 1. (**Normality Axiom**) $M\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2. (**Duality Axiom**) $M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event $\Lambda \in L$.
Axiom 3. (**Subadditivity Axiom**) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$M\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leqslant \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

Definition 2.1. (Liu [12]) *Let Γ be a nonempty set, let L be a σ-algebra over Γ, and let M be an uncertain measure. Then the triplet (Γ, L, M) is called an uncertainty space.*

Uncertain variable is mainly used to model the uncertain quantities. A formal definition was given by Liu [12] as follows.

Definition 2.2. (Liu [12]) *An uncertain variable is a function ξ from an uncertainty space (Γ, L, M) to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B.*

Theorem 2.1. (Liu [15]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(\xi_1, \xi_2, \ldots, \xi_n)$ is strictly increasing with respect to ξ_1, ξ_2,..., ξ_m and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \ldots, \xi_n$, then*

$$M\{f(\xi_1, \xi_2, \ldots, \xi_n) \leqslant 0\} \geqslant \alpha$$

if and only if

$$f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right) \leqslant 0.$$

Example 2.1. *Assume that $\xi \sim Z(a, b, c)$ is a zigzag uncertain variable where a, b, and c are real numbers with $a < b < c$. Its uncertainty distribution $\Phi(x)$ and inverse uncertainty distribution $\Phi^{-1}(\alpha)$ are given by*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leqslant a \\ (x-a)/2(b-a), & \text{if } a \leqslant x \leqslant b \\ (x+c-2b)/2(c-b), & \text{if } b \leqslant x \leqslant c \\ 1, & \text{if } x \geqslant c, \end{cases}$$

and

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\ (2-2\alpha)b + (2\alpha-1)c, & \text{if } \alpha \geqslant 0.5, \end{cases}$$

respectively.

2.2. **Classic maximum weight independent set problem.** Let $G = (V, E)$ be a connected, undirected and simple graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E = \{(v_i, v_j) \mid v_i \in V, v_j \in V, i < j\}$. An independent set in a graph G is a subset I of V such that no two vertices in this subset are adjacent. There are two basic problems in the area of independent set problem. One is maximum independent set problem (MISP), and the other is maximum weight independent set problem (MWISP). The objective of the MISP is to find a maximum subset of V such that there is no edge between any two vertices in the subset. Assume that each vertex is associated with a positive weight, and let the weight of an independent set be the total weight of its vertices. The key issue of the MWISP is to find an independent set with a largest total weight. In fact, the maximum independent set and maximum weight independent set all belong to the independent set, but they are different. We give an example to illustrate this point.

Example 2.2. *Assume that there are 5 vertices in the graph G, the weight of each vertex is shown in Figure 1.*

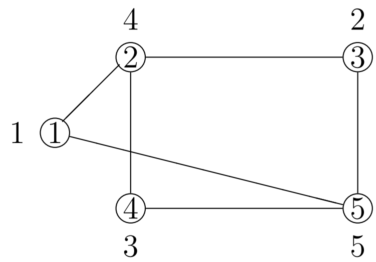

FIGURE 1. Weighted graph G

Obviously, the independent set $I = \{v_1, v_3, v_4\}$ is a maximum independent set, but is not a maximum weight independent set. Similarly, the independent set $I = \{v_2, v_5\}$ is a maximum weight independent set, but is not a maximum independent set.

In this paper, we are mainly concerned with the problem of finding a maximum weight independent set in the vertex weighted graph G, that is an independent set with maximum weight. We write w_i to denote the weight of each vertex v_i, and all the weights are presented by a vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$.

Let $I \subseteq V$, and

$$x_i = \begin{cases} 1, & \text{if } i \in I \\ 0, & \text{otherwise.} \end{cases}$$

Then any independent set of the graph G can be denoted by an n-dimensional binary vector $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ over $\{0, 1\}^n$. Similarly, any n-dimensional binary vector $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ over $\{0, 1\}^n$ corresponds to an independent set of G. Then the decision variable set is denoted by $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$.

Let the weight of an independent set be the total weight of its vertices. We formulate the weight of independent set I as follows

$$W(I) = \sum_{v_i \in V} w_i x_i.$$

According to the formal definition of the independent set, no vertices are adjacent in an independent set. So a subset I of V is an independent set if and only if

$$x_i + x_j \leqslant 1, \quad \forall (v_i, v_j) \in E.$$

Thus we construct the following mathematical model for MWISP as follows:

$$\begin{cases} \max\limits_{\boldsymbol{x}} \quad \sum\limits_{v_i \in V} w_i x_i \\ \text{subject to:} \\ \qquad x_i + x_j \leqslant 1, \qquad \forall (v_i, v_j) \in E \\ \qquad x_i \in \{0, 1\}, \qquad \forall v_i \in V. \end{cases} \tag{1}$$

3. **Uncertain Mathematical Model.** The model (1) assumes that the vertex weights are crisp values. In practice, however, the weight of vertex usually is not a constant number. However, in many cases, we have a lack of observed data or observed data is invalid because unexpected events have occurred. Then, one problem is naturally produced, that is how can we deal with this kind of indeterminacy factors in the maximum weight independent set problem? In this situation, the vertex weight data can only be obtained from the decision-makers' empirical estimation in a practical way. As mentioned before, uncertainty theory provides a new tool to deal with uncertain information, especially subjective or empirical data. Hence, in this paper, we assume that vertex weights w_i are uncertain variables ξ_i, $i = 1, 2, \ldots, n$, respectively. Without loss of generality, we also assume that vertex weights are nonnegative and independent uncertain variables. All the weights are presented by $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$. Then the maximum weight independent set problem becomes uncertain maximum weight independent set problem, which is denoted by UMWISP for short. For UMWISP discussed in this paper, the independent weight $W(I)$ is still an uncertain variable. As is known for all of us, we cannot rank the uncertain variables directly. Then we should employ some rank criteria to select the optimal independent set.

Definition 3.1. *Let $G = (V, E)$ be an undirected and simple graph with uncertain vertex weights. An independent set I^* is called the α-maximum independent set among uncertain weight independent set if*

$$\max\left\{\overline{W} \mid M\left\{W(I^*) \geqslant \overline{W}\right\} \geqslant \alpha\right\} \geqslant \max\left\{\overline{W} \mid M\{W(I) \geqslant \overline{W}\} \geqslant \alpha\right\}$$

holds for any independent set I of G, where α is a predetermined confidence level given by decision maker.

Now we apply the belief degree constrained programming model to uncertain maximum weight independent set problem, and present an α-maximum weight independent set model:

$$\begin{cases} \max\limits_{\boldsymbol{x}} \quad \overline{W} \\ \text{subject to:} \\ \qquad M\left\{\sum\limits_{v_i \in V} \xi_i x_i \geqslant \overline{W}\right\} \geqslant \alpha \\ \qquad x_i + x_j \leqslant 1, \qquad \forall (v_i, v_j) \in E \\ \qquad x_i \in \{0, 1\}, \qquad \forall v_i \in V, \end{cases} \tag{2}$$

where α is the predetermined confidence level given by decision maker.

Theorem 3.1. *Let $G = (V, E)$ be an undirected and simple graph with uncertain vertex weights, and ξ_i independent uncertain variables with regular uncertainty distributions Φ_i,*

$i = 1, 2, \ldots, n$, respectively. Then the model (2) is equivalent to the following model

$$\begin{cases} \max\limits_{\boldsymbol{x}} \quad \sum\limits_{v_i \in V} x_i \Phi_i^{-1}(1-\alpha) \\ \text{subject to:} \\ \qquad x_i + x_j \leqslant 1, \qquad \forall (v_i, v_j) \in E \\ \qquad x_i \in \{0, 1\}, \qquad \forall v_i \in V, \end{cases} \tag{3}$$

where Φ_i^{-1} is the inverse uncertainty distributions of ξ_i.

Proof: It follows from Theorem 2.1 that

$$M\left\{\sum_{v_i \in V} x_i \xi_i \geqslant \overline{W}\right\} \geqslant \alpha$$

is equivalent to

$$\sum_{v_i \in V} x_i \Phi_i^{-1}(1-\alpha) \geqslant \overline{W}.$$

Then we can easily prove that the model (2) can be equivalently transformed into the following deterministic model:

$$\begin{cases} \max\limits_{\boldsymbol{x}} \quad \overline{W} \\ \text{subject to:} \\ \qquad \sum\limits_{v_i \in V} x_i \Phi_i^{-1}(1-\alpha) \geqslant \overline{W} \\ \qquad x_i + x_j \leqslant 1, \qquad \forall (v_i, v_j) \in E \\ \qquad x_i \in \{0, 1\}, \qquad \forall v_i \in V. \end{cases} \tag{4}$$

Clearly, Model (4) is equivalent to Model (3). The theorem is proved. □

Definition 3.2. (Liu [13]) *Let ξ be an uncertain variable, $\alpha \in (0, 1]$, and $r \in R$. Then*

$$\xi_{\sup}(\alpha) = \sup\{r \mid M\{\xi \geqslant r\} \geqslant \alpha\}$$

is called the α-optimistic value to ξ.

Lemma 3.1. (Liu [13]) *Let ξ be an uncertain variable. Then $\xi_{\sup}(\alpha)$ is a decreasing function with respect to α.*

Since the objective of Model (3) is a function of the parameter α, the relationship between the optimal objective and the parameter α should be investigated. The following theorem will answer this question.

Theorem 3.2. *Let α_1 and α_2 be two parameters, and W_1 and W_2 the corresponding optimal objectives of Model (5). If $\alpha_1 \geqslant \alpha_2$, then we have $W_1 \leqslant W_2$.*

Proof: Let D be the feasible domain of Model (3). According to Lemma 3.1, the optimistic value is a decreasing function with respect to α. Then we have

$$W_1 = \sum_{v_i \in V} x_i \Phi_i^{-1}(1-\alpha_1) \leqslant \sum_{v_i \in V} x_i \Phi_i^{-1}(1-\alpha_1) = W_2.$$

Thus, the proof is completed. □

4. **Numerical Example.** In this section, we give a numerical example to illustrate the applicability of the proposed models as mentioned above. Suppose that there are eight new projects for a company. We can draw a conflict graph G, where the vertex represents a desirable project, and an edge joining two vertices v_i and v_j indicates that at most one of v_i and v_j can be selected. Now, the task for the decision maker is to select as many of the desirable projects as possible while not selecting two conflicting projects. At the beginning of this task, the decision maker needs to obtain the basic data, such as mercantile rate of return, profitability index, and static investment payback period. However, due to economic reasons or technical difficulties, the decision maker always cannot get these data exactly. For this condition, the usual way is to obtain the uncertain data by means of experience evaluation or expert advice. The graph G is shown in Figure 2. Assume that all vertex weights are zigzag uncertain variables ξ_i, which are listed in Table 1.

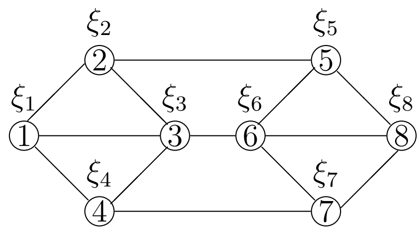

FIGURE 2. Uncertain weighted graph G

TABLE 1. List of ξ_i

ξ_i	Φ_i	ξ_i	Φ_i
ξ_1	$Z(2,3,4)$	ξ_5	$Z(5,6,7)$
ξ_2	$Z(3,4,5)$	ξ_6	$Z(2,3,6)$
ξ_3	$Z(2,3,5)$	ξ_7	$Z(5,7,9)$
ξ_4	$Z(2,5,7)$	ξ_8	$Z(5,7,8)$

When α=0.9, we can calculate $\Phi_i^{-1}(0.1)$ for each ξ_i. The values are listed in Table 2.

TABLE 2. List of $\Phi_i^{-1}(0.1)$

ξ_i	$\Phi_i^{-1}(0.1)$	ξ_i	$\Phi_i^{-1}(0.1)$
ξ_1	2.2	ξ_5	5.2
ξ_2	3.2	ξ_6	2.2
ξ_3	2.2	ξ_7	5.4
ξ_4	2.6	ξ_8	5.4

According to the model (3), the 0.9-maximum weight independent set problem can be formulated as follows:

$$\begin{cases} \max\limits_{\boldsymbol{x}} \quad \overline{W} \\ \text{subject to:} \\ \qquad M\left\{\sum\limits_{i=1}^{8} \xi_i x_i \geqslant \overline{W}\right\} \geqslant 0.9 \\ \qquad x_i + x_j \leqslant 1, \quad i,j=1,2,\ldots,8, i<j \\ \qquad x_i \in \{0,1\}, \quad i=1,2,\ldots,8. \end{cases} \tag{5}$$

According to Theorem 3.1, the model (5) is equivalent to the deterministic programming model:

$$\begin{cases} \max\limits_{\boldsymbol{x}} \quad \sum\limits_{i=1}^{8} x_i \Phi_i^{-1}(0.1) \\ \text{subject to:} \\ \qquad x_i + x_j \leqslant 1, \quad i,j = 1,2,\ldots,8, i<j \\ \qquad x_i \in \{0,1\}, \quad i = 1,2,\ldots,8. \end{cases} \tag{6}$$

The optimal solution of the model (6) can be obtained as $x^* = (0,0,1,0,1,0,1,0)^T$ by using the values listed in Table 2 and the mathematical software (e.g., LINGO). The optimal value of the objective is equal to 12.8. Since the objective of the belief degree constrained programming model is a function of the parameter α, the sensitivity of the optimal objective can be investigated with respect to different parameters. The following predetermined confidence levels are selected to test the sensitivity: $\alpha = 0.1$, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9. By choosing different confidence levels α, we obtain Table 3. From Table 3, we find that α has an effect on the optimal solutions, and the total weight of the maximum weight independent set increases as the confidence level α decreases, which just coincides with Theorem 3.2.

TABLE 3. List of α-maximum weighted independent set

α	optimal solution x^*	α-maximum weighted independent set	maximum weight
0.9	$(0,0,1,0,1,0,1,0)^T$	$\{3,5,7\}$	12.8
0.8	$(0,0,1,0,1,0,1,0)^T$	$\{3,5,7\}$	13.6
0.7	$(0,0,1,0,1,0,1,0)^T$	$\{3,5,7\}$	14.4
0.6	$(0,0,1,0,1,0,1,0)^T$	$\{3,5,7\}$	15.2
0.5	$(0,1,0,1,0,0,0,1)^T$	$\{2,4,8\}$	16
0.4	$(1,0,0,0,1,0,1,0)^T$	$\{1,5,7\}$	16.8
0.3	$(1,0,0,0,1,0,1,0)^T$	$\{1,5,7\}$	17.6
0.2	$(0,0,1,0,1,0,1,0)^T$	$\{3,5,7\}$	19
0.1	$(1,0,0,0,1,0,1,0)^T$	$\{1,5,7\}$	19.2

5. Conclusions.

In the real-life application, we usually encounter some uncertain factors due to lack of observed data about the unknown state of nature. Different from other researches in indeterminacy environment, this paper investigated the uncertain maximum weight independent set problem based on uncertainty theory, and the vertex weights were assumed to be uncertain variables. The main contributions include the following three aspects. (i) In order to deal with uncertain factors in maximum weight independent set problem, uncertainty theory was introduced into the problem under the light of uncertain environment. (ii) A belief degree constrained programming model for maximum weight independent set problem in uncertain environment was proposed. We found that the maximum weight of belief degree constrained programming model was actually decreasing with respect to the predetermined confidence level. (iii) The illustrative example was given to show the application of the proposed model.

The proposed uncertain programming models in this paper have their unique advantages, that is, they can be transformed into their corresponding deterministic forms. In spite of these advantages, a few issues need to be addressed in the future. For example, we can continue to study the maximum bisection problem in uncertain environment within the axiomatic framework of uncertainty theory. In addition, the current work can be extend to the uncertain random environment, where uncertainty and randomness coexist in a system.

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