ICIC Express Letters
Part B: Applications
Volume **8**, Number **5**, May **2017**

ICIC International ©2017 ISSN 2185-2766
pp. **803–810**

A STAGE-STRUCTURED PREY-PREDATOR MODEL WITH CANNIBALISM IN THE PREY AND A FIXED IMPULSE FOR INTEGRATED PEST MANAGEMENT

Changcheng Xiang, Juanzi Jiang and Yi Yang

Key Laboratory of Biologic Resource Protection and Utilization of Hubei Province
Hubei University for Nationalities
No. 39, Xueyuan Road, Enshi 445000, P. R. China
xcc7426681@126.com

Received October 2016; accepted January 2017

Abstract. *In this paper, a stage-structured prey-predator model with cannibalism in the prey is constructed and investigated for the purpose of integrated pest management. To address how such factors influence successful pest control, hybrid impulsive control of pesticide sprays and natural enemy releases were proposed and analyzed. The pest-eradication periodical solution of the model is globally stable. However, when the impulsive period is greater than some threshold, the pest population and natural enemy population can coexist. Multiple attractors from which the pest population oscillates with different amplitudes can coexist for a wide range of parameters. Numerical simulations imply the switch-like transitions by varying the numbers of natural enemies released.*
Keywords: A stage-structured prey-predator model, Cannibalism, Integrated pest management, Global stability, Permanence

1. **Introduction.** In this paper, we consider the stage-structured prey-predator model with cannibalism for the purpose of integrated pest management. Recently, more and more scholars have studied the stage-structured prey-predator models in the pest management; many methods have been proposed to control the pest population, such as spraying pesticides and releasing natural enemies. Some papers are devoted to formulating models to study it [1, 2]. As is well known, impulsive state feedback control strategies are widely used in real world problems. For example, in the pest management, control measures will be taken when the pest population reaches the Economic Threshold (the pest population density at which control measures should be undertaken to prevent an increasing pest population from reaching the economic injury level which is the lowest pest population density that will cause economic damage) [3, 4, 5].

The basic stage-structured prey-predator model with cannibalism [6] is

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = b_1x_2(t) - cx_1(t) - b_2x_1(t)x_2(t) - m_1x_1(t), \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = cx_1(t) + sb_2x_1(t)x_2(t) - m_2x_2(t) - \dfrac{a_1x_2(t)y(t)}{1+x_2(t)}, \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = \dfrac{a_2x_2(t)y(t)}{1+x_2(t)} - m_3y(t), \end{cases} \tag{1}$$

where $x_1(t)$, $x_2(t)$ and y represent the densities of immature prey, mature prey and predator, respectively. We give the following assumptions for the model. b_1, c and m_1 represent the birth rate, recruitment rate and death rate of the immature prey population, respectively; b_2 denotes the cannibalism attacking rate of the mature prey population; m_2 denotes the death rate of the mature prey population; s is the conversion rate of the immature prey into the mature prey due to cannibalism, and according to the biological meaning, it is easy to know $s < 1$; the parameter m_3 and a_1 represent the death rate and attacking rate of the predator population, respectively. a_2 is the rate of conversion of nutrients into the reproduction of the predator.

Let $(x_1^*(t), x_2^*(t), y^*(t))$ be any solution of system (1), and it is easy to know that

$$\begin{cases} x_1^*(t) = \dfrac{b_1 m_3}{(a_1 - m_3)(c + m_1) + b_2 m_3}, \\ x_2^*(t) = \dfrac{m_3}{(a_1 - m_3)}, \\ y^*(t) = \dfrac{c x_1(t) + s b_2 x_1(t) x_2(t) - m_2 x_2(t)}{m_3}, \end{cases} \tag{2}$$

it is obvious that if $a_1 > m_3$, then $x_1^*(t) > 0$, $x_2^*(t) > 0$, $y^*(t) > 0$, which means that system (1) has and only has a positive equilibrium point and system (1) is permanent.

In this article, we consider the stage-structured predator-prey model with cannibalism for pest management. In the second section, we proposed the combination of biological (natural enemies), cultural (catching), and chemical (killing) tactics that eradicates the pest to extinction, and prove the locally asymptotical stablility, globally asymptotical stablility for the the pest-eradication solution. Finally, we numerically studied the system with respect to bifurcation diagram for implusive T.

2. **Biological Integrated Control Strategy.**

2.1. **The model with an impulsive effect at fixed moment.**

A constant periodic releasing for the predator and a proportional periodic impulsive catching or poisoning for the pest populations at fixed moment are described as follows

$$\begin{cases} \left.\begin{aligned} \frac{dx_1(t)}{dt} &= b_1 x_2(t) - c x_1(t) - b_2 x_1(t) x_2(t) - m_1 x_1(t) \\ \frac{dx_2(t)}{dt} &= c x_1(t) + s b_2 x_1(t) x_2(t) - m_2 x_2(t) - \frac{a_1 x_2(t) y(t)}{1 + x_2(t)} \\ \frac{dy(t)}{dt} &= \frac{a_2 x_2(t) y(t)}{1 + x_2(t)} - m_3 y(t) \end{aligned}\right\} & t \neq nT, \\ \triangle x_1(t) = -p_1 x_1(t),\ \triangle x_2(t) = -p_2 x_2(t),\ \triangle y(t) = \tau & t = nT, \end{cases} \tag{3}$$

where $\triangle x_1(t) = x_1(t^+) - x_1(t)$, $\triangle x_2(t) = x_2(t^+) - x_2(t)$, $\triangle y(t) = y(t^+) - y(t)$, and T is the period of the impulsive effect. It is easy to find that all solutions of system (3) remain positive if the initial conditions are positive and $0 \le p_1 \le 1$, $0 \le p_2 \le 1$, $\tau \ge 0$.

In the section, some notations, definitions and lemmas which are useful for stating and proving main results are given. Let $R_+ = (0, +\infty)$, $R_+^3 = \{X = (x_1, x_2, x_3) \in R^3 : x_1, x_2, x_3 > 0\}$. Denote $f = (f_1, f_2, f_3)$ the mapping defined by the right-hand side of system (3). Let $V_0 = (V : R_+ \times R_+^3 \to R_+)$, V_0 is continuous on $(nT, (n+1)T] \times R_+^3$ and $\lim_{(t,y)\to(nT^+,x)} V(t,y) = V(nT^+, x)$ exists.

Definition 2.1. *For $V \in V_0$ and $(t, x) \in (nT, (n+1)T] \times R_+^3$, the upper right Dini derivative of $V(t, x)$ with respect to the impulsive differential system* (3) *is defined as*

$$D^+V(t,x) = \lim_{h\to 0^+} sup \frac{1}{h} \left[V(t+h, x + hf(t,x)) - V(t,x) \right],$$

the solution of system (3) *denoted by $Z(t) = (x_1(t), x_2(t), y(t)) : R_+ \to R_+^3$, $Z(t)$ is continuous on $(nT, (n+1)T]$, $n \in N$, and $Z(n\tau^+) = \lim_{t\to n\tau^+} Z(t)$ exists. Obviously, the global existence and the uniqueness of solution of system (3) are guaranteed by the smoothness properties of f, for details see* [7].

Lemma 2.1. *Let $V \in V_0$, and assume that*

$$\begin{cases} D^+V(t,x) \le g(t, V(t,x)), & t \ne nT, \\ V(t, x(t^+)) \le \psi_n(V(t, x(t))), & t = nT, \end{cases} \tag{4}$$

where $g : R_+ \times R_+ \mapsto R$ satisfies: g is continuous in $(nT, (n+1)T] \times R_+$ and for $x \in R_+$, $n \in N$, $\lim_{(t,y)\to(nT^+,x)} g(t,y) = g(nT^+, x)$ exists. And $\psi_n : R_+ \mapsto R_+$ is non-decreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation $u'(t) = g(t, u(t))$, $t \ne nT$, $\triangle u(t) = \psi_n(u(t))$, $t = nT$, $u(0^+) = u_0 \ge x_0$. Then $V(0^+, x_0) \le u_0$ implies that $V(t, x(t)) \le r(t)$ for $t \ge 0$ where $x(t)$ is any solution of system (3). For convenience, we give some basic properties of the following system

$$\begin{cases} u'(t) = -wu(t), & t \ne nT, \\ \triangle u(t) = \tau, & t = nT, \\ u(0^+) = u_0 \ge 0. \end{cases} \tag{5}$$

Lemma 2.2. *System* (5) *has a positive periodic solution $u^*(t)$, and for every solution $u(t)$ of system* (5) *with positive initial value $u(0^+)$, $|u(t) - u^*(t)| \to 0$ as $t \to \infty$, where $u^*(t) = \frac{\tau \exp[-w(t-nT)]}{1-\exp(-wT)}$, $nT < t \le (n+1)T$, $n \in N$, $u(0^+) = \frac{\tau}{1-\exp(-wT)}$.*

Theorem 2.1. *There exists a positive constant L, such that $x_1(t) \le L$, $x_2(t) \le L$, $y(t) \le L$ for each solution $(x_1(t), x_2(t), y(t))$ of system (3) with positive initial values, where t is large enough.*

Proof: Define a function V such that $V(t) = x_1(t) + x_2(t) + y(t)$. Then $V(t) \in V_0$, and the upper right derivative of $V(t)$ is described as

$$\begin{aligned} D^+V_t|_{(1,1)} + mV(t) &= b_1x_2(t) - (1-s)b_2x_1(t)x_2(t) - (m_1 - m)x_1(t) \\ &\quad -(m_2 - m)x_2(t) - (m_3 - m)y(t) - \frac{(a_1 - a_2)x_2(t)y(t)}{1 + x_2(t)} \le l_0, \end{aligned} \tag{6}$$

where $m = \min\{m_1, m_2, m_3\}$, we see that

$$V(nT^+) = V(nT) + \tau, \tag{7}$$

according to Lemma 2.1, we have

$$\begin{aligned} V(t) &= V(0)e^{-mt} + \textstyle\int_0^t le^{-m(t-s)}ds + \sum_{0<nT<t} me^{-m(t-nT)} \\ &\le V(0)e^{-mt} + \frac{l(1-e^{-mt})}{m} \to \frac{l}{m} + \frac{\tau e^{mT}}{e^{mT} - 1},\ t \to \infty, \end{aligned} \tag{8}$$

consequently, by the definition of $V(t)$, we obtain each solution of system with positive initial values is uniformly ultimately bounded above. □

2.2. **Stability of pest-eradication solution.** The solution of system (3) corresponds to $x_1(t) = 0$, $x_2(t) = 0$ which is called pest-eradication solution.

Theorem 2.2. *The pest-eradication periodic solution $(0, 0, y^*(t))$ is locally asymptotically stable.*

Proof: For y, according to Lemma 2.2, we obtain the form of the unique positive periodical solution with pulses: $y^*(t) = \frac{\tau \exp[-m_3(t-nT)]}{1-\exp(-m_3T)}$, $nT \le t \le (n+1)T$, $n \in N$ with initial value $y^*(0) = \frac{\tau}{1-\exp(-m_3T)}$. Now, we study the stability of the pest-eradication solution. And the Jacobi matrix $H(t)$ at $(0, 0, y^*(t))$ and $I(t)$ are as follows

$$H(t) = \begin{pmatrix} -(c+m_1) & b_1 & 0 \\ c & -(m_2 + a_1y^*(t)) & 0 \\ 0 & a_2y^*(t) & -m_3 \end{pmatrix},\ I(t) = \begin{pmatrix} (1-p_1) & 0 & 0 \\ 0 & (1-p_2) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we can calculate the monodromy matrix $M(t)$

$$M(t) = I(t)\exp\left(\int_0^T H(t)dt\right) = \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{pmatrix}, \tag{9}$$

where $\mu_1 = (1-p_1)\exp(-(c+m_1)T)$, $\mu_2 = (1-p_2)\exp\left(\int_0^T (m_2 + a_1 y^*)dt\right)$, $\mu_3 = \exp(-m_3 T)$. It is obvious that $|\mu_1| < 1$, $|\mu_3| < 1$, so the stability of $(0,0,y^*(t))$ is decided by μ_2; if $|\mu_2| < 1$, it would be stable. That is, $\mu_2 = (1-p_2)\exp\left(\int_0^T (m_2 + a_1 y^*)dt\right) < 1$, we calculate that when $T > -\frac{1}{m_2}\left(\ln\frac{1}{(1-p_2)} + \frac{a_1\tau}{m_3}\right)$, the pest-eradication solution is locally stable. In fact, as periodic time T is always a positive constant, in this situation, the periodic solution $(0,0,y^*(t))$ is locally asymptotically stable without any conditions. □

Further, we would prove that the pest-eradication solution is globally asymptotically stable.

Theorem 2.3. *If* $T > \left\{-\frac{1}{c+m_1}\ln\frac{1}{1-p_1}, -\frac{1}{m_2}\ln\frac{1}{1-p_2}\right\}$, *then the pest-eradication solution of system* (3) *is globally asymptotically stable.*

Proof: From the first equation of system (3), we easily get that $x_1'(t) \geq -(c+m_1)x_1(t)$ considering the following equation

$$\begin{cases} z_1'(t) = -(c+m_1)z_1(t), & t \neq nT, \\ \triangle z_1(nT) = -p_1 z_1(nT), & t = nT, \end{cases} \tag{10}$$

by Lemma 2.2 and system (10), we obtain that

$$z_1 = z_1(nT^+)\exp\left(\int_{nT}^t (c+m_1)dt\right) = z_1\left(nT^+\right)\exp((-c-m_1)(t-nT)), \tag{11}$$

by the second equation of system (10), we get the difference equation

$$z_1((n+1)T^+) = (1-p_1)z_1(nT^+)\exp((-c-m_1)(t-nT)), \tag{12}$$

here, we obtain that $T > -\frac{1}{c+m_1}\ln\frac{1}{1-p_1}$ when $(1-p_1)\exp((-c-m_1)T) > 1$, then $z_1(nT^+) = (1-p_1)\exp((-c-m_1)T)z_1(0^+) \to 0$ as $n \to \infty$. So system (10) has a globally asymptotically stable periodic solution $z_1^*(t) = 0$.

Next, we prove that $y(t) \to y^*(t)$ as $t \to \infty$. From the third equation of system (3), note that $y'(t) \geq -m_3 y(t)$. Consider the following impulsive differential equation

$$\begin{cases} z_3'(t) = -m_3 z_3(t), & t \neq nT, \\ \triangle z_3'(t) = \tau, & t = nT, \\ z_3(0^+) = y(0^+) \geq 0, \end{cases} \tag{13}$$

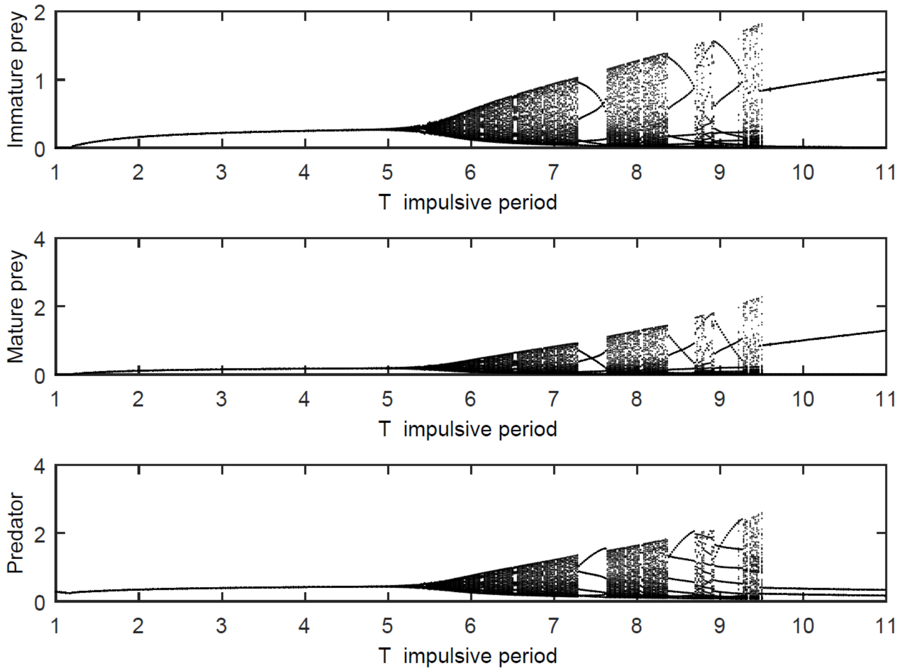
by Lemma 2.1 we get that $y(t) \geq z_3(t) > y^*(t) - \varepsilon$. For a sufficiently small $\varepsilon > 0$, we have

$$\begin{cases} y(t) \leq \varepsilon - m_3 y(t), & t \neq nT, \\ \triangle y(t) = \tau, & t = nT, \end{cases} \tag{14}$$

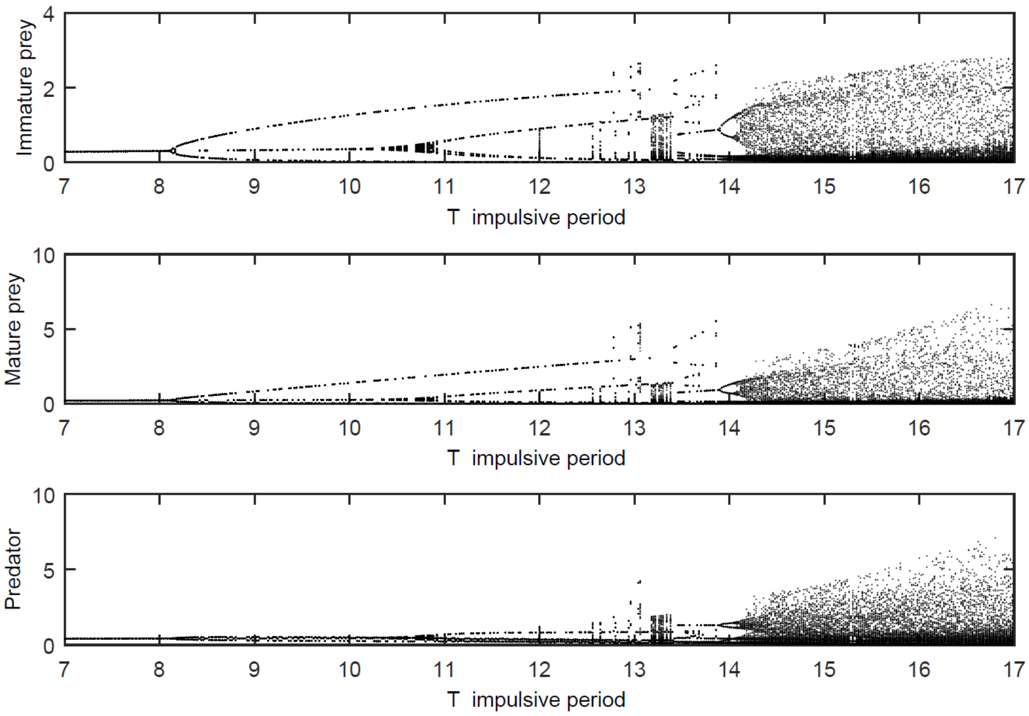
by Lemma 2.2 we get that

$$y^*(t) = \frac{\tau\exp[-m_3(t-nT)]}{1-\exp(-m_3 T)},\ nT < t \leq (n+1)T,\ n \in N, \tag{15}$$

we have $y(t) < y^* + \varepsilon$. Then combining $y(t) > y^* - \varepsilon$ and $y(t) < y^* + \varepsilon$, we obtain $y^* - \varepsilon < y(t) < y^* + \varepsilon$ for t large enough. Let $\varepsilon \to 0$, and we get $y(t) \to y^*(t)$ as $t \to \infty$. Therefore, the periodic solution $(0,0,y^*(t))$ is globally asymptotically stable. □



(a) The parameters are fixed as $p_1 = 0.3$, $p_2 = 0.2$, $\tau = 0.1$, $b_1 = 2.5$, $c = 1$, $b_2 = 0.65$, $m_1 = 0.5$, $s = 0.5$, $m_2 = 0.3$, $a_1 = 3$, $a_2 = 2$, $m_3 = 0.3$.



(b) The parameters are fixed as $p_1 = 0.7$, $p_2 = 0.8$, $\tau = 0.2$, $b_1 = 2.5$, $c = 1$, $b_2 = 0.65$, $m_1 = 0.5$, $s = 0.5$, $m_2 = 0.3$, $a_1 = 3$, $a_2 = 2$, $m_3 = 0.3$.

FIGURE 1. Bifurcation diagram for impulsive period T

Theorem 2.4. *If* $T < \left\{-\frac{1}{c+m_1}\ln\frac{1}{1-p_1}, -\frac{1}{m_2}\ln\frac{1}{1-p_2}\right\}$, *then system* (3) *is uniform permanent.*

Proof: As Theorem 2.1 states, there exists a positive constant L, such that $x_1(t)$, $x_2(t)$, $y(t) \le L$; we only need to find a positive constant H, such that $x_1(t)$, $x_2(t)$, $y(t) \ge H$ for t large enough. we can easily get that $x_1(t)$, $x_2(t)$, $y(t)$ satisfy the following three systems respectively,

$$\begin{cases} x_1'(t) = -(c+m_1) * x_1(t), & t \neq nT, \\ \triangle x_1(nT) = -p_1 x_1(nT), & t = nT, \\ x_2'(t) = -m_2 * x_2(t), & t \neq nT, \\ \triangle x_2(nT) = -p_2 x_2(nT), & t = nT, \\ y'(t) = -m_3 * y(t), & t \neq nT, \\ \triangle y(nT) = \tau, & t = nT. \end{cases} \tag{16}$$

By Lemma 2.1, Lemma 2.2 and Theorem 2.3, for any sufficient small positive number ε, when $T < \left\{-\frac{1}{c+m_1} \ln \frac{1}{1-p_1}, -\frac{1}{m_2} \ln \frac{1}{1-p_2}\right\}$, the following three inequalities hold

$$\begin{cases} x_1(nT^+) \geq (1-p_1) * \exp(-(c+m_1)T) x_1(0^+) - \varepsilon = h_1, \\ x_2(nT^+) \geq (1-p_2) * \exp(-m_2 T) x_2(0^+) - \varepsilon = h_2, \\ y(t) > y^*(t) - \varepsilon = h, \end{cases} \tag{17}$$

so $x_1(t)$, $x_2(t)$, $y(t)$ are ultimately positively bounded below and every solution of system (3) will eventually enter and remind in region $\Omega = \{(x_1(t), x_2(t), y(t)) : x_1 \geq h_1, x_2 \geq h_2, y \geq h, x_1(t) \leq L, x_2(t) \leq L, y(t) \leq L\}$. □

3. **Numeric Analysis and Discussion.** In order to understand how these control method affect the final state of the pest population, we numerically studied the model with respect to bifurcation parameter of impulsive period T.

In Figure 1(a), we studied the impact of impulsive period on the complexity of system (3), according to numerical results, There are very complicated dynamic behavior, such as period-doubling cascade, symmetry-breaking pitchfork bifurcation, and chaos. When T lies in [4, 5], it seems to keep stable. As T increase from 5 to 5.4, the periodic attractor appears, subsequently, a cascade of period-doubling bifurcations leading to chaos. However, when we consider the significance of initial values, we can find some different conclusions from Figure 1(b). We can find that there are three different solutions of system when $T \in (12.8, 13.7)$.

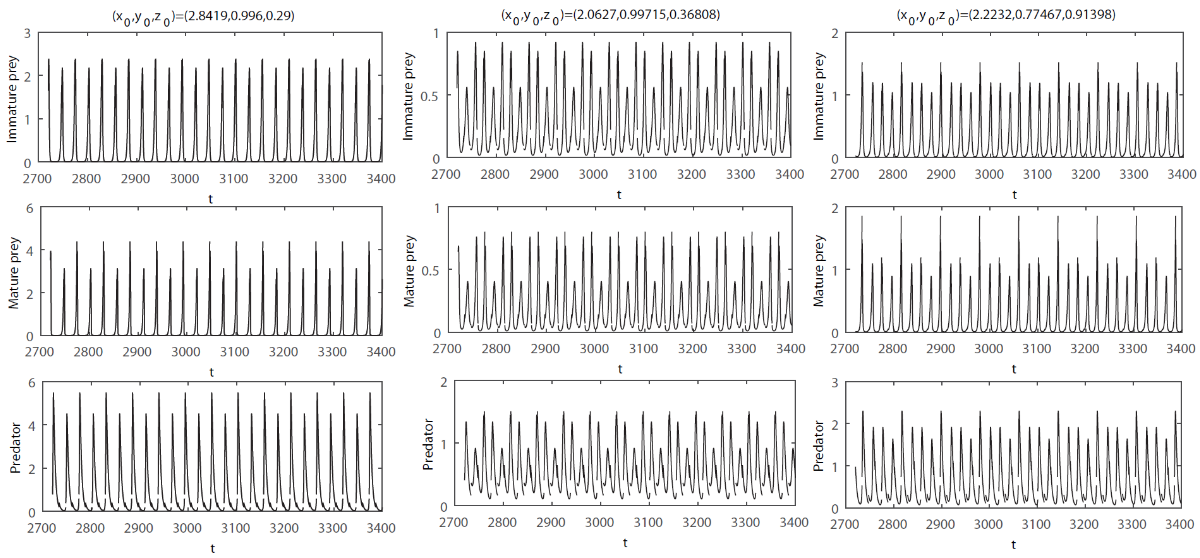


FIGURE 2. Three coexisting attractors of $T = 13.6$, the other parameters fixed as $p_1 = 0.7$, $p_2 = 0.8$, $\tau = 0.2$, $b_1 = 2.5$, $c = 1$, $b_2 = 0.65$, $m_1 = 0.5$, $s = 0.5$, $m_2 = 0.3$, $a_1 = 3$, $a_2 = 2$, $m_3 = 0.3$

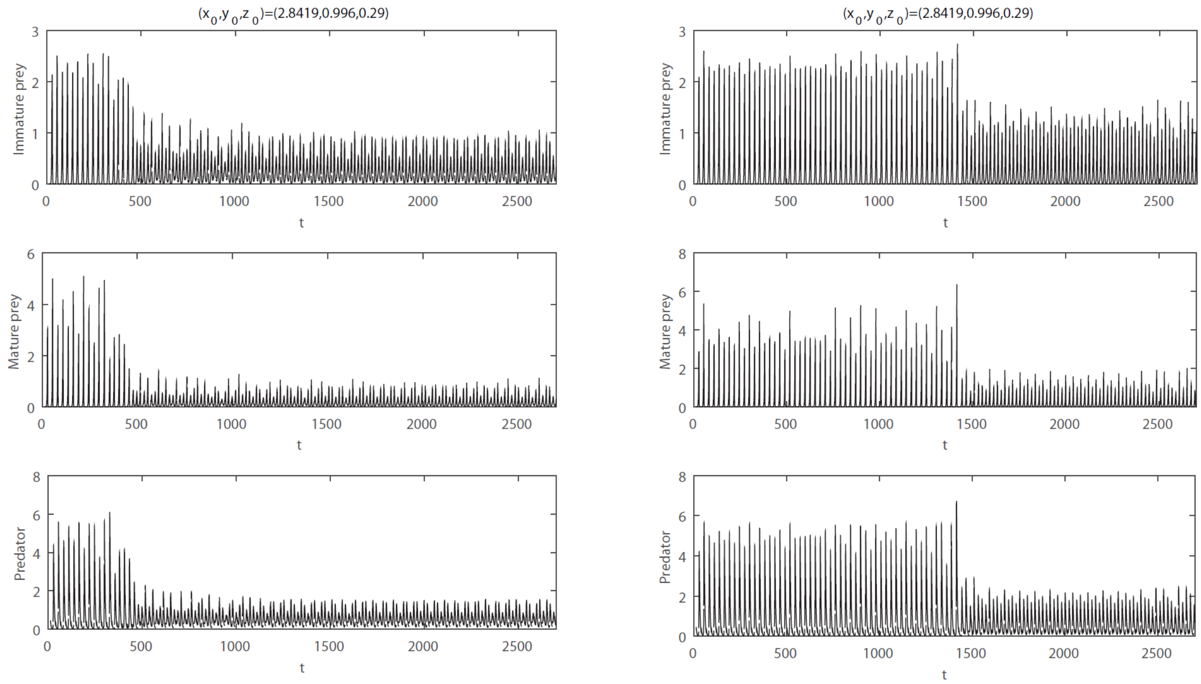


FIGURE 3. Attractors' switch-like behavior with small random pertubation on parameter τ, the other parameters as shown in Figure 2

In order to give some detailed explanations of coexistence attractor in Figure 1(b), we can find that different initial values affect the coexistence of attractors. When $T = 13.6$, there exist three different types of impulse state as shown in Figure 2. That is to say, initial values are the key factor in the integrate pest management strategy. Furthermore, when small random perturbations are introduced in the parameter τ of the system (3), numerical simulations imply that those solutions can switch to another attractor with smaller amplitude at a random time (Figure 3). This implies that impulsive effect makes the dynamic behavior of system more complicated. A lot results suggest a new and appropriate approach in pest control, and we hope that these results could bring a bright insight to pest management.

4. **Conclusion.** This work focused entirely on the stage-structured predator-prey model with impulsive effects and the temporal interactions of an insect pest and its natural enemy. The pest-eradication periodical solution of the model is global stable. Numerical simulations imply the switch-like transitions by varying the numbers of natural enemies released.

Acknowledgment. This work is partially supported by the Natural Science Foundation of Hubei Province (2015CFB264, Changcheng Xiang).

REFERENCES

[1] X. Liu and L. Chen, Complex dynamics of holling type II lotka-volterra predator-prey system with impulsive perturbations on the predator, *Chaos Solitons & Fractals*, vol.16, no.2, pp.311-320, 2003.
[2] X. Song and Z. Xiang, The prey-dependent consumption two-prey one-predator models with stage structure for the predator and impulsive effects, *Journal of Theoretical Biology*, vol.242, no.3, pp.683-698, 2006.
[3] S. Tang, G. Tang and R. A. Cheke, Optimum timing for integrated pest management: Modelling rates of pesticide application and natural enemy releases, *Journal of Theoretical Biology*, vol.264, no.2, pp.623-638, 2010.
[4] W. Gao and S. Tang, The effects of impulsive releasing methods of natural enemies on pest control and dynamical complexity, *Nonlinear Analysis Hybrid Systems*, vol.5, no.3, pp.540-553, 2011.

[5] Z. Xiang, S. Tang, C. Xiang and J. Wu, On impulsive pest control using integrated intervention strategies, *Applied Mathematics & Computation*, vol.269, no.3, pp.930-946, 2015.

[6] L. Zhang et al., Rich dynamic of a stage-structured prey-predator model with cannibalism and periodic attacking rate, *Communications in Nonlinear Science & Numerical Simulation*, vol.15, no.12, pp.4029-4040, 2010.

[7] D. Bainov and P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, CRC Press, 1993.