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A SMOOTHING METHOD FOR ABSOLUTE VALUE EQUATION BASED ON A NEW SMOOTHING FUNCTION

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Abstract. *In this paper, firstly, we propose a new smoothing function which approximates the absolute value function. Secondly, using this function, we recast the absolute value equation (AVE) as a smoothing system, and then apply a smoothing Newton method to solving the system of equations. Under some assumptions, we get that the algorithm is globally and locally quadratically convergent. Finally, we report the numerical results.*
Keywords: Absolute value equation, Smoothing function, Smoothing algorithm

1. **Introduction.** The absolute value equation (AVE) (Rohn [1]) is to find a point $x \in IR^n$ such that

$$Ax + B|x| = b, \tag{1}$$

where $A \in IR^{n\times n}$, $B \in IR^{n\times n}$, $b \in IR^n$ and $|\cdot|$ means absolute value. A special form of (1) is

$$Ax - |x| = b. \tag{2}$$

The equation of (1) or (2) has attracted much attention in the literature, for example, Mangasarian [2, 3, 4, 5, 6], Rohn [7, 8, 9, 10], Prokopyev [11], Mangasarian and Meyer [12], Jiang and Zhang [13], Caccetta et al. [14].

The papers mentioned above talk about issues as below:

- The existence of the solution;
- How to convert AVE to LCP;
- Solvability of the AVE and the number of solutions;
- The numerical algorithms for the AVE.

Jiang and Zhang [13] firstly considered the function $\phi_p(\mu, a) = \sqrt[p]{|a|^p + |\mu|^p}$, which approximates the absolute value $|a|$, and then presented smoothing Newton method and got good numerical results. Inspired by this idea, we construct a smoothing function $\varphi_p(\mu, a)$ to approximate $|a|$. Compared with the function ϕ_p in [13], our function φ_p is a piecewise function. For any fixed $\mu > 0$, from calculating the distance of two functions, we find that

$$\min_{a\in IR}\{\phi_p(\mu, a) - |a|\} > 0,\ \min_{a\in IR}\{\varphi_p(\mu, a) - |a|\} = 0,$$
$$\max_{a\in IR}\{\phi_p(\mu, a) - |a|\} = \mu,\ \max_{a\in IR}\{\varphi_p(\mu, a) - |a|\} < \mu.$$

In fact, the mathematical structures of functions ϕ_p and φ_p indicate that

$$\phi_p(\mu, a) > \varphi_p(\mu, a) > |a|.$$

From all the above, we see that φ_p is the one which best approximates the function $|a|$. Then, we reformulate AVE (1) as a smoothing equation. Like [13], we also use the smoothing-type method to solve the problem AVE (1). Furthermore, we obtain the same convergence results with [13], which has the global and local quadratic convergence properties. At the end of this paper, we provide some numerical experiments.

2. **A Smoothing Function and Its Properties.** For any $p \in [2, +\infty)$, define the function $\varphi_p : R_+ \times R \to R$ by

$$\varphi_p(\mu, t) = \begin{cases} t & t \geq \mu, \\ 2^{\frac{p-1}{p}} \sqrt[p]{|t|^p + \mu^p} - \mu & -\mu < t < \mu, \\ -t & t \leq -\mu. \end{cases} \tag{3}$$

This function is different from the function $\phi_p(\mu, a) = \sqrt[p]{|a|^p + |\mu|^p}$ used in [13]. Our function φ_p possesses the following properties.

Lemma 2.1. *Let φ_p be defined as in (3). Then,*
(i) *φ_p is continuously differentiable on $IR_{++} \times IR$.*
(ii) $\big|\varphi_p(\mu, t) - |t|\big| < 4\mu$.

Proof: (i) First, we verify that

$$\frac{\partial \varphi_p(\mu, t)}{\partial \mu} = \begin{cases} 0 & t \geq \mu, \\ 2^{\frac{p-1}{p}} \mu^{p-1} (|t|^p + \mu^p)^{\frac{1-p}{p}} - 1 & -\mu < t < \mu, \\ 0 & t \leq -\mu, \end{cases}$$

$$\frac{\partial \varphi_p(\mu, t)}{\partial t} = \begin{cases} 1 & t \geq \mu, \\ 2^{\frac{p-1}{p}} \mathrm{sgn}(t) |t|^{p-1} (|t|^p + \mu^p)^{\frac{1-p}{p}} & -\mu < t < \mu, \\ -1 & t \leq -\mu. \end{cases}$$

Then, it is clear to see that $\lim\limits_{t \to \mu} \frac{\partial \varphi_p(\mu, t)}{\partial t} = 1$ and $\lim\limits_{t \to -\mu} \frac{\partial \varphi_p(\mu, t)}{\partial t} = -1$, because

$$\lim_{t \to \mu^+} \frac{\partial \varphi_p(\mu, t)}{\partial t} = 1, \quad \lim_{t \to \mu^-} \frac{\partial \varphi_p(\mu, t)}{\partial t} = \lim_{t \to \mu^-} 2^{\frac{p-1}{p}} |t|^{p-1} (|t|^p + \mu^p)^{\frac{1-p}{p}} = 1,$$

and

$$\lim_{t \to -\mu^-} \frac{\partial \varphi_p(\mu, t)}{\partial t} = -1, \quad \lim_{t \to -\mu^+} \frac{\partial \varphi_p(\mu, t)}{\partial t} = \lim_{t \to -\mu^+} 2^{\frac{p-1}{p}} \mathrm{sgn}(t) |t|^{p-1} (|t|^p + \mu^p)^{\frac{1-p}{p}} = -1.$$

Hence, $\frac{\partial \varphi_p(\mu, t)}{\partial t}$ is continuous.

Similarly, we have

$$\lim_{t \to \mu^+} \frac{\partial \varphi_p(\mu, t)}{\partial \mu} = \lim_{t \to \mu^-} \frac{\partial \varphi_p(\mu, t)}{\partial \mu} = 0$$

and

$$\lim_{t \to -\mu^+} \frac{\partial \varphi_p(\mu, t)}{\partial \mu} = \lim_{t \to -\mu^-} \frac{\partial \varphi_p(\mu, t)}{\partial \mu} = 0.$$

Hence, $\frac{\partial \varphi_p(\mu, t)}{\partial \mu}$ is continuous.

The analysis mentioned above implies that φ_p is continuously differentiable at point $(\mu, t) \in IR_{++} \times IR$.

(ii) By a routine computation, we obtain that

$$\big|\varphi_p(\mu, t) - |t|\big| = \begin{cases} 0 & t \geq \mu, \\ \left|2^{\frac{p-1}{p}} \sqrt[p]{|t|^p + \mu^p} - \mu - |t|\right| & -\mu < t < \mu, \\ 0 & t \leq -\mu. \end{cases}$$

Since $p \geq 2$ and $|t| < \mu$, we know

$$\left|2^{\frac{p-1}{p}} \sqrt[p]{|t|^p + \mu^p} - \mu - |t|\right| \leq 2^{\frac{p-1}{p}} \sqrt[p]{|t|^p + \mu^p} + \mu + |t| < 4\mu.$$

Therefore, we obtain

$$\left|\varphi_p(\mu, t) - |t|\right| < 4\mu. \qquad \square$$

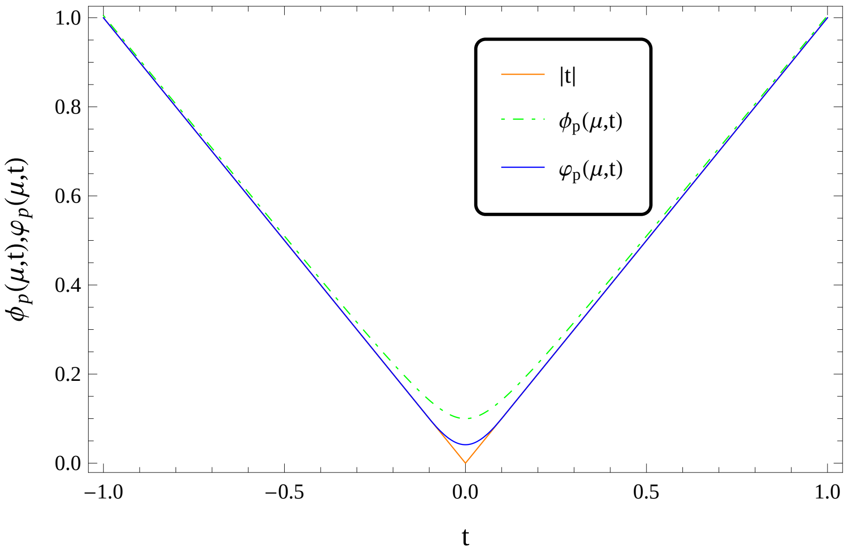

FIGURE 1. $\mu = 0.1$, $p = 2$

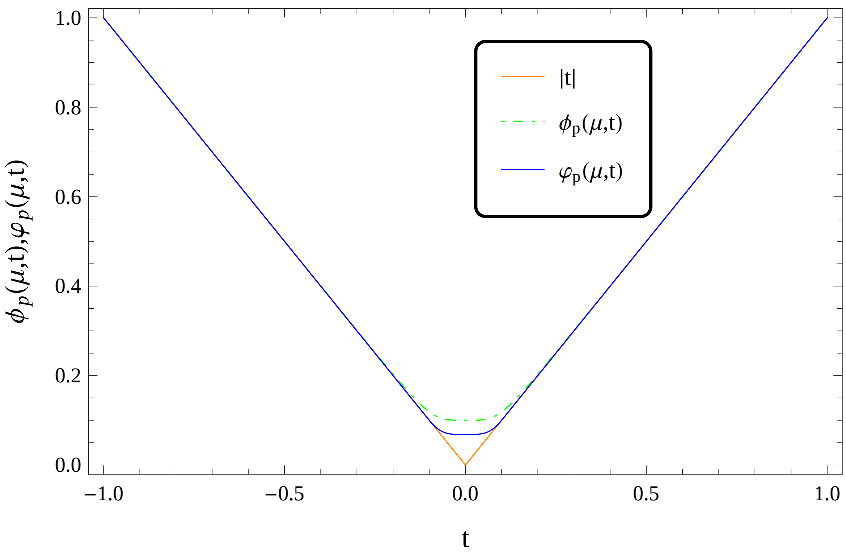

FIGURE 2. $\mu = 0.1$, $p = 4$

From figures, we find that φ_p will be close to $|x|$ when $\mu \downarrow 0$. In fact, for any $\mu > 0$, we obtain

$$\lim_{|t|\to\infty} |\varphi_p(\mu, t) - |t|| = 0, \quad \lim_{|t|\to\infty} |\phi_p(\mu, t) - |t|| = 0$$

and

$$\max_{t\in IR} |\varphi_p(\mu, t) - |t|| = |\varphi_p(\mu, 0)| = \left|2^{\frac{p-1}{p}} \sqrt[p]{\mu^p} - \mu\right| = \left(2^{\frac{p-1}{p}} - 1\right)\mu,$$
$$\max_{t\in IR} |\phi_p(\mu, t) - |t|| = |\phi_p(\mu, 0)| = \mu.$$

As $p \in [2, +\infty)$, $\sqrt{2} < 2^{\frac{p-1}{p}} < 2$, this gives that $\left(2^{\frac{p-1}{p}} - 1\right)\mu < \mu$. The above expressions indicate that φ_p is the function which best approximates the function $|t|$.

For any $p \geq 2$, let

$$\psi_p(\mu, x) = (\varphi_p(\mu, x_1), \cdots, \varphi_p(\mu, x_n))^T, \quad \forall\ (\mu, x) \in IR_+ \times IR^n. \tag{4}$$

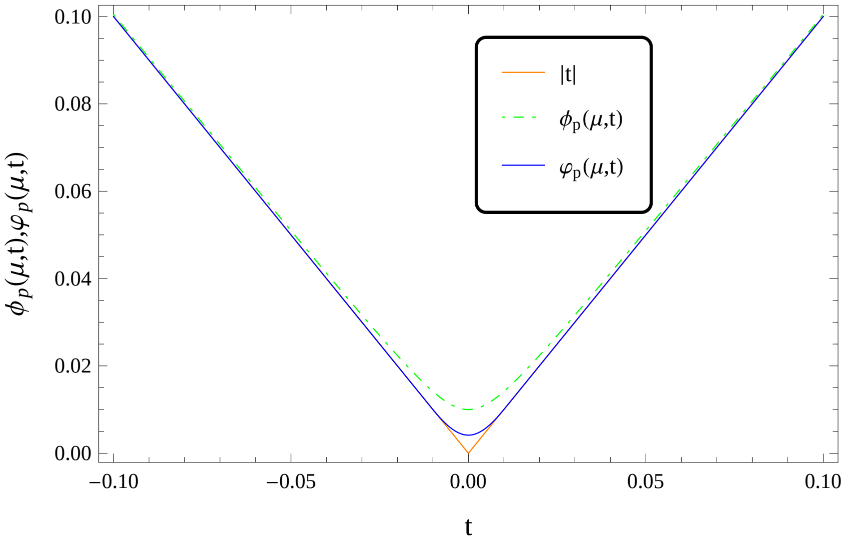

FIGURE 3. $\mu = 0.01$, $p = 2$

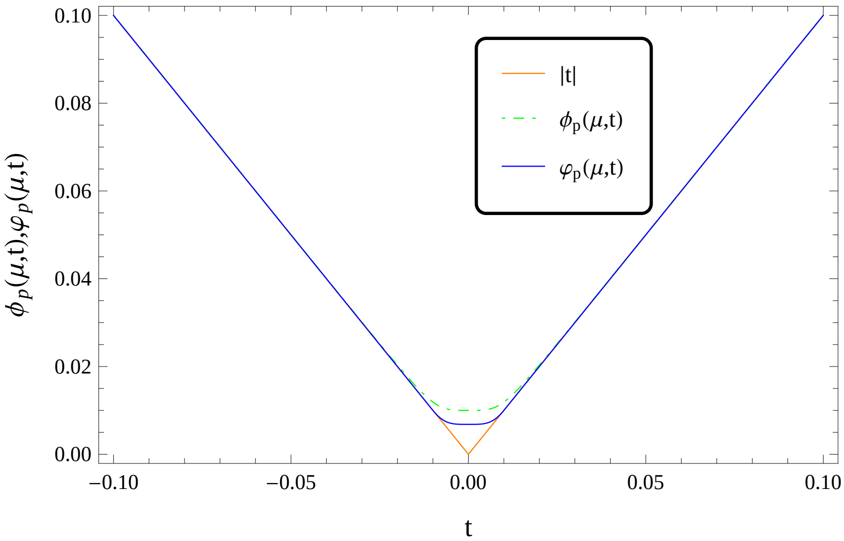

FIGURE 4. $\mu = 0.01$, $p = 4$

Define the function $G_p : IR_+ \times IR^n \to IR^{n+1}$ by

$$G_p(\mu, x) = \begin{pmatrix} \mu \\ Ax + B\psi_p(\mu, x) - b \end{pmatrix}. \tag{5}$$

Lemma 2.2. *Let $\psi_p(\mu, x)$ be defined by (4). Then*
(i) *$\psi_p(\mu, x)$ is a regular smoothing function of $|x|$.*
(ii) *For $p = 2$, $\psi_p(\mu, x)$ approximates $|x|$ quadratically.*

Proof: (i) According Lemma 2.1 (ii), we find that

$$\left\|\psi_p(\mu, x) - |x|\right\| \le \left|\varphi_p(\mu, x_1) - |x_1|\right| + \cdots + \left|\varphi_p(\mu, x_n) - |x_n|\right| < 4n\mu.$$

(ii) From Lemma 2.6 (ii) in [14], part (ii) is clear. □

Lemma 2.3. *Let $G_p(\mu, x)$ be defined by (5). Then*
(i) *$G_p(\mu, x) = 0 \Leftrightarrow x$ solves AVE (1).*
(ii) *G_p is continuously differentiable on $IR_{++} \times IR^n$ with*

$$G'_p(\mu, x) = \begin{pmatrix} 1 & 0 \\ B\frac{\partial \psi_p(\mu,x)}{\partial \mu} & A + B\frac{\partial \psi_p(\mu,x)}{\partial x} \end{pmatrix},$$

where

$$\frac{\partial \psi_p(\mu,x)}{\partial \mu} = \left(\frac{\partial \varphi_p(\mu,x_1)}{\partial \mu}, \cdots, \frac{\partial \varphi_p(\mu,x_n)}{\partial \mu}\right)^T,$$

$$\frac{\partial \psi_p(\mu,x)}{\partial x} = \text{diag}\left(\frac{\partial \varphi_p(\mu,x_1)}{\partial x_1}, \cdots, \frac{\partial \varphi_p(\mu,x_n)}{\partial x_n}\right).$$

Proof: To prove the result (i), we recall the AVE (1) and Lemma 2.2 (i). Therefore, part (i) holds. In addition, part (ii) is straightforward. □

Now, we state the following assumption.

Assumption 2.1. *The minimal singular value of the matrix A is strictly greater than the maximal singular value of the matrix B.*

Lemma 2.4. ([13]) *The AVE (1) is uniquely solvable for any $b \in IR^n$ if Assumption 2.1 is satisfied.*

3. **The Smoothing Algorithm and Its Convergence Properties.** In this section, instead of solving the AVE (1), one may solve $G_p(\mu,x) = 0$. Hence, we focus on the smoothing algorithm discussed in [13]. By making $\mu \downarrow 0$, a solution of AVE (1) can be founded.

Algorithm

Step 0. Choose σ, $\delta \in (0,1)$, $p \in [2,+\infty)$, $\mu_0 > 0$, $x^0 \in IR^n$. Set $z^0 = (\mu_0, x^0)$. Let $e^0 = (1,0) \in IR \times IR^n$. Select $\beta > 1$ such that $\min\left\{1, \left\|G_p(z^0)\right\|\right\} \le \sqrt{\beta\mu_0}$. Set $k = 0$.

Step 1. If $\left\|G_p(z^k)\right\| = 0$, stop. Otherwise, compute $\gamma_k = \min\left\{1, \left\|G_p(z^k)\right\|\right\}$.

Step 2. Get $\triangle z^k = \left(\triangle\mu_k, \triangle x^k\right)$ by

$$G_p'(z^k)\triangle z^k = -G_p(z^k) + \frac{1}{\beta}\gamma_k^2 e^0. \tag{6}$$

Step 3. Set α_k to be the maximum of the values $1, \delta, \delta^2, \cdots$ satisfying

$$\left\|G_p(z^k + \alpha_k\triangle z^k)\right\| \le \left[1 - \sigma\left(1 - \frac{1}{\beta}\right)\alpha_k\right]\left\|G_p(z^k)\right\|. \tag{7}$$

Step 4. Let $z^{k+1} = \left(\mu_{k+1}, x^{k+1}\right) = z^k + \alpha_k\triangle z^k$. Replace $k = k+1$ and go to Step 1.

According to Assumption 2.1, following the same arguments as in [13], it is obvious that the convergence properties of our algorithm are similar to that in [13]. Therefore, we omit the proof.

Theorem 3.1. *Let $p \in [2,+\infty)$ and Assumption 2.1 be satisfied. Suppose that the sequence $\{z^k\}$ is generated by our algorithm with $z^* = (\mu_*, x^*)$ being an accumulation point. Then, we have*

(i) $\lim\limits_{k\to\infty} z^k = z^*$.

(ii) $\left\|z^{k+1} - z^*\right\| = O\left(\left\|z^k - z^*\right\|^2\right)$ *and* $\mu_{k+1} = O\left(\mu_k^2\right)$.

4. **Numerical Results.** In what follows, we report our numerical results for solving AVE (1) with the smoothing function chosen as the φ_p and ϕ_p respectively. Our codes are finished by MATLAB. During the testing, we set the parameters in our algorithm as $\sigma = 0.0001$, $\delta = 0.2$, $\mu_0 = 0.0001$. The error tolerance is $\varepsilon = 1.0e-6$. The algorithm was terminated once $\left\|H(z^k)\right\| < \varepsilon$ or $\left|\left\|H(z^k)\right\| - \left\|H(z^{k+1})\right\|\right| < \varepsilon$.

The testing problem: First, choose randomly $A \in IR^{200\times200}$ and $B \in IR^{200\times200}$ from $[-10,10]$. Second, let $[U,S,V] = svd(A)$. If $\min\{S(i,i) : 1 \le i \le n\} = 0$, then

$A = V(S+0.01I)V$, where I is the identity matrix. Third, set $A = \dfrac{\lambda_{\max}(B^TB)+0.01}{\lambda_{\min}(A^TA)}A$. Finally, choose randomly $u \in IR^{200}$ from $[-1,1]$, and then let $b = Au + B|u|$.

Three cases of the starting point x^0 were considered. (i) Random x^0 from $[-1,1]$. (ii) $x^0 = (1,1,\cdots,1)^T$. (iii) $x^0 = (0,0,\cdots,0)^T$.

In our experiments, we select $p = 2,3,4,5,6,7,8$ and every case is implemented ten times. Tables 1 and 2 contain our results. In our tables, Max, Min, It, T, Va and Fa

TABLE 1. Iterations for function φ_p

p		2	3	4	5	6	7	8
Case (i)	Max	4	7	4	4	4	4	4
	Min	3	3	3	3	3	3	3
	It	3.1	3.5	3.1	3.3	3.1	3.1	3.1
	T	0.0134	0.0174	0.0152	0.0155	0.0152	0.0146	0.0152
	Va	7.85E-07	1.01E-06	1.83E-07	9.01E-07	3.79E-07	3.54E-07	5.26E-07
	Fa	0	0	1	0	0	0	0
Case (ii)	Max	4	4	4	10	4	4	4
	Min	3	3	3	3	3	3	3
	It	3.2	3.1	3.4	3.9	3.1	3.1	3.1
	T	0.0130	0.0146	0.0159	0.0182	0.0151	0.0141	0.0144
	Va	6.35E-07	2.57E-07	8.27E-07	9.68E-06	3.37E-07	3.22E-07	2.59E-07
	Fa	0	1	0	0	0	0	0
Case (iii)	Max	4	5	4	10	4	3	11
	Min	3	3	3	3	3	3	3
	It	3.2	3.3	3.2	4	3.1	3	4
	T	0.0131	0.0180	0.0152	0.0191	0.0148	0.0141	0.0185
	Va	6.95E-07	7.05E-07	2.36E-06	6.76E-06	1.14E-06	2.04E-07	2.75E-06
	Fa	0	0	0	0	0	0	1

TABLE 2. Iterations for function ϕ_p

p		2	3	4	5	6	7	8
Case (i)	Max	11	8	5	15	4	4	5
	Min	6	4	3	3	3	3	3
	It	9.7	4.9	3.7	4.9	3.2	3.1	3.4
	T	0.0375	0.0156	0.0166	0.0159	0.0147	0.0110	0.0164
	Va	2.45E-07	2.80E-07	6.14E-06	1.72E-06	4.19E-07	4.13E-07	1.54E-06
	Fa	0	0	0	0	0	0	1
Case (ii)	Max	11	8	6	4	4	5	8
	Min	5	4	3	3	3	3	3
	It	8.2	5.6	4	3.3	3.3	3.3	3.6
	T	0.0208	0.0237	0.0137	0.0112	0.0113	0.0115	0.0123
	Va	4.62E-07	9.89E-07	1.28E-06	4.89E-07	4.66E-07	3.66E-07	3.34E-06
	Fa	0	1	0	0	0	0	0
Case (iii)	Max	12	6	5	7	6	4	4
	Min	4	4	3	3	3	3	3
	It	7.2	4.3	3.8	4	3.7	3.2	3.4
	T	0.0184	0.0138	0.0125	0.0138	0.0123	0.0117	0.0114
	Va	4.32E-07	1.93E-07	1.03E-06	7.49E-07	1.80E-06	2.61E-07	4.60E-07
	Fa	0	0	0	0	0	0	0

mean the maximal number of iterations, the minimal number of iterations, the average value of the iterations, the average value of the CPU time in seconds, the average value of $\|H(z^k)\|$ when algorithm stops and the total number of algorithm fails, respectively.

From Tables 1 and 2, we see that the numerical performance based on our function φ_p is better when $p = 2$ and $p = 3$. On the whole, our function can solve the testing problems in few iterations and little CPU time. Therefore, the smoothing method based on the function φ_p is effective for solving AVE.

5. **Conclusions.** In this paper, we discuss a new smoothing function. By using this function, we reformulate the AVE as a smoothing system. Then, we apply a smoothing Newton algorithm to solving it. We find that φ_p is the one which best approximates the function $|t|$, and it is also the best when the numerical results are taken into account. Therefore, for future work, comparison with other types of algorithms is desirable.

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