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OPTIMAL PROGRESSIVE FIRST-FAILURE CENSORING PLANS FOR THE EXPONENTIATED WEIBULL PRODUCTS WITH COST CONSTRAINT

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Abstract. *In this study, the lifetime of the product has the exponentiated Weibull distribution with the three parameters (γ, β, λ). Next, we design three computational algorithms for the method of D-optimality p-order quantile estimator, and survival function to determine the optimum solutions of the number of testing group (n) and the number of test in group (k) under the progressive first-failure censoring plan with restrictions relating to the cost of the life testing, respectively. Finally, one numerical example and sensitivity analysis are taken into consideration to illustrate the proposed approach.*
Keywords: Exponentiated Weibull distribution, Progressive first-failure censoring plan, Maximum likelihood estimator, The cost of the life testing, Sensitivity analysis

1. **Introduction.** Suppose that m is the number of failures observed before termination and n independent groups with k items within each group are put in a life test. $\tilde{r}_1$ groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure (say X_1) has occurred, $\tilde{r}_2$ groups and the group in which the second failure is observed are randomly removed from the test as soon as the second failure (say X_2) has occurred, and finally $\tilde{r}_m$ $(m \leq n)$ groups and the group in which the mth failure is observed are randomly removed from the test as soon as the mth failure (say X_m) has occurred. Then $X_1 < X_2 < \cdots < X_m$ are called the progressive first-failure censored order statistics with censoring scheme $\tilde{r} = (\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_m)$ (also see [1]).

The exponentiated Weibull distribution with the probability density function (p.d.f.) $f(x)$ and the cumulative distribution function (c.d.f.) $F(x)$ are as follows respectively:

$$f(x) = \frac{\gamma\beta}{\lambda}\left[1 - \exp\left(-(x/\lambda)^{\beta}\right)\right]^{\gamma-1} \exp\left(-(x/\lambda)^{\beta}\right)(x/\lambda)^{\beta-1}, \tag{1}$$

and

$$F(x) = \left[1 - \exp\left(-(x/\lambda)^{\beta}\right)\right]^{\gamma}, \ x > 0, \ \gamma > 0, \ \beta > 0, \ \lambda > 0, \tag{2}$$

where the shape of the exponentiated Weibull distribution is determined by the shape parameters γ and β. The effect of the scale parameter λ is to only stretch out the plot. Thus, the shape parameters γ and β are more important than the scale parameter λ (also see [2]). Hence, in this paper, let $\lambda = \lambda_0$ be the known (or fixed) scale parameter.

In the design of reliability sampling plans with progressive first-failure censored data, one needs to decide the number of groups, the number of test units in each group, and the critical point. One practical problem arising from designing a reliability sampling plan is the cost of the experiment. However, in the literature, many related researchers [3-5] considered the cost restriction when designing sampling plans. Wu and Huang [4] proposed an approach to establishing reliability sampling plans which minimize three different objective functions under the constraint of total cost of experiment and given consumer's and producer's risks. Attia and Assar [3] proposed optimal progressive group-censoring plans for Weibull distribution in presence of cost constraint with the unknown shape parameter β and the known scale parameter λ. Lee et al. [5] designed an optimal life test based on the progressive type I group censoring plan under the Weibull distribution. The purpose of this study is to explore the optimal number of groups and the optimal number of test units in each group in conducting a reliability sampling plan. In this study, the lifetime of the product has the exponentiated Weibull distribution with the three parameters (γ, β, λ). The shape parameters γ and β are more important than the scale parameter λ in the exponentiated Weibull distribution. Hence the present study proposes a 'shape-first' fitting approach to fit the shape parameters γ and β under the fixed scale parameter λ. We design three computational algorithms for the method of D-optimality, p-order quantile estimator, and survival function to determine the optimum solutions of the number of testing group (n) and the number of test in group (k) under the progressive first-failure censoring plan with restrictions relating to the cost of the life testing, respectively. This study is organized as follows. Section 2 presents the maximum likelihood estimator of the parameters of exponentiated Weibull distribution with the progressively first-failure censoring plan. Section 3 presents the estimated asymptotic variance matrix of maximum likelihood estimators with the method of D-optimality, the estimated asymptotic variance of p-order quantile estimator, and the estimated asymptotic variance of survival function estimator. Section 4 studies the design of reliability sampling plans to determine the number of groups and the number of test units in each group, and then sets up a reliability sampling plan with cost consideration. Section 5 applies the proposed approach to one numerical example. The sensitivity analysis is investigated in Section 6. Some conclusions and discussions are given in Section 7.

2. **The Maximum Likelihood Estimator of the Parameters.** Let X denote the lifetime of a product and X has an exponentiated Weibull distribution with the p.d.f. $f(x)$ as (1) and c.d.f. $F(x)$ as (2). $X_1, X_2, \cdots, X_m$ are the progressive first-failure censored order statistics from the exponentiated Weibull distribution with censoring scheme $\tilde{r} = (\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_m)$. The likelihood function is

$$\widetilde{L}(\gamma, \beta, \lambda) = c \cdot k^m \prod_{i=1}^{m} \left\{ \gamma\beta\lambda^{-1} \left(\tilde{h}_i\right)^{\gamma-1} \left(1 - \tilde{h}_i\right) (x_i/\lambda)^{\beta-1} \left[1 - \tilde{h}_i^{\gamma}\right]^{k(\tilde{r}_i+1)-1} \right\}, \quad (3)$$

where $c = n(n - \tilde{r}_1 - 1)(n - \tilde{r}_1 - \tilde{r}_2 - 2) \cdots (n - \tilde{r}_1 - \tilde{r}_2 - \cdots - \tilde{r}_{m-1} - m + 1)$, $\tilde{h}_i = 1 - \exp\left(-(x_i/\lambda)^{\beta}\right)$.

The differentiation of the log-likelihood function $\ln \widetilde{L}(\gamma, \beta, \lambda)$ with respect to γ, β and λ yields

$$\frac{\partial \ln \widetilde{L}(\gamma, \beta, \lambda)}{\partial \gamma} = \frac{m}{\gamma} + \sum_{i=1}^{m} \ln \tilde{h}_i - \sum_{i=1}^{m} \left\{ [k(\tilde{r}_i + 1) - 1] \left(\frac{\tilde{h}_i^{\gamma} \cdot \ln \tilde{h}_i}{1 - \tilde{h}_i^{\gamma}} \right) \right\}, \quad (4)$$

$$\frac{\partial \ln \widetilde{L}(\gamma, \beta, \lambda)}{\partial \beta} = \frac{m}{\beta} + (\gamma - 1) \sum_{i=1}^{m} \left(\frac{1}{\tilde{h}_i} \frac{\partial \tilde{h}_i}{\partial \beta} \right) - \sum_{i=1}^{m} \left(\frac{1}{1 - \tilde{h}_i} \frac{\partial \tilde{h}_i}{\partial \beta} \right) + \sum_{i=1}^{m} \ln(x_i) - m \ln \lambda - \sum_{i=1}^{m} \left\{ [k(\tilde{r}_i + 1) - 1] \left(\frac{\gamma \tilde{h}_i^{\gamma-1}}{1 - \tilde{h}_i^{\gamma}} \frac{\partial \tilde{h}_i}{\partial \beta} \right) \right\}, \quad (5)$$

and

$$\frac{\partial \ln \tilde{L}(\gamma,\beta,\lambda)}{\partial \lambda} = (\gamma-1)\sum_{i=1}^{m}\left(\frac{1}{\tilde{h}_i}\cdot\frac{\partial \tilde{h}_i}{\partial \lambda}\right) + \frac{\beta}{\lambda^{\beta+1}}\sum_{i=1}^{m}\left(x_i^{\beta}\right) - \frac{m\beta}{\lambda} - \sum_{i=1}^{m}\left\{[k(\tilde{r}_i+1)-1]\left(\frac{\gamma\tilde{h}_i^{\gamma-1}}{1-\tilde{h}_i^{\gamma}}\frac{\partial \tilde{h}_i}{\partial \lambda}\right)\right\}, \tag{6}$$

where $\frac{\partial \tilde{h}_i}{\partial \beta} = (x_i/\lambda)^{\beta}\ln(x_i/\lambda)\exp\left(-(x_i/\lambda)^{\beta}\right)$ and $\frac{\partial \tilde{h}_i}{\partial \lambda} = -\beta x_i^{\beta}\lambda^{-\beta-1}\exp\left(-(x_i/\lambda)^{\beta}\right)$.

The maximum likelihood estimators $\hat{\gamma}$, $\hat{\beta}$ and $\hat{\lambda}$ can be derived by solving the nonlinear equations $\frac{\partial \ln \tilde{L}(\gamma,\beta,\lambda)}{\partial \gamma} = 0$, $\frac{\partial \ln \tilde{L}(\gamma,\beta,\lambda)}{\partial \beta} = 0$ and $\frac{\partial \ln \tilde{L}(\gamma,\beta,\lambda)}{\partial \lambda} = 0$ with Compaq Visual Fortran version 6.6 [6] and IMSL subroutine NEQNF.

3. **The Estimated Asymptotic Variance of Estimators.** The estimated asymptotic covariance matrix $I^{-1}(\hat{\gamma},\hat{\beta})$ of the maximum likelihood estimators $\hat{\gamma}$ and $\hat{\beta}$, the estimated asymptotic variance of p-order quantile estimator, and the estimated asymptotic variance of survival function estimator can be obtained in large sample theory under the unknown shape parameters γ, β and the fixed scale parameter $\lambda = \lambda_0$ with λ_0 being known.

By the log-likelihood function $\ln \tilde{L}(\gamma,\beta,\lambda)$, and the fixed scale parameter $\lambda=\lambda_0$ with λ_0 being known, we have

$$\frac{\partial^2 \ln \tilde{L}(\gamma,\beta,\lambda_0)}{\partial \gamma^2} = \frac{-m}{\gamma^2} - \sum_{i=1}^{m}\left\{[k(\tilde{r}_i+1)-1]\left[\tilde{h}_i^{\gamma}\left(\ln\tilde{h}_i\right)^2\left(1-\tilde{h}_i^{\gamma}\right) + \left(\tilde{h}_i^{\gamma}\ln\tilde{h}_i\right)^2\right]\left(1-\tilde{h}_i^{\gamma}\right)^{-2}\right\}. \tag{7}$$

$$\begin{aligned}\frac{\partial^2 \ln \tilde{L}(\gamma,\beta,\lambda_0)}{\partial \beta^2} = & -\frac{m}{\beta^2} + (\gamma-1)\sum_{i=1}^{m}\left[\frac{-1}{\tilde{h}_i^2}\left(\frac{\partial \tilde{h}_i}{\partial \beta}\right)^2 + \frac{1}{\tilde{h}_i}\frac{\partial^2 \tilde{h}_i}{\partial \beta^2}\right] \\ & - \sum_{i=1}^{m}\left(\left[\frac{\partial^2 \tilde{h}_i}{\partial \beta^2}\left(1-\tilde{h}_i\right) + \left(\frac{\partial \tilde{h}_i}{\partial \beta}\right)^2\right]\left(1-\tilde{h}_i^{\gamma}\right)^{-2}\right) \\ & - \sum_{i=1}^{m}\left\{[k(\tilde{r}_i+1)-1]\left[\left[\gamma(\gamma-1)\tilde{h}_i^{\gamma-2}\frac{\partial \tilde{h}_i}{\partial \beta}\left(1-\tilde{h}_i^{\gamma}\right)\right.\right.\right. \\ & \left.\left.\left. +\gamma^2\left(\tilde{h}_i^{\gamma-1}\right)^2\frac{\partial \tilde{h}_i}{\partial \beta}\right]\left(1-\tilde{h}_i^{\gamma}\right)^{-2}\cdot\frac{\partial \tilde{h}_i}{\partial \beta} + \left(\frac{\gamma\tilde{h}_i^{\gamma-1}}{1-\tilde{h}_i^{\gamma}}\cdot\frac{\partial^2 \tilde{h}_i}{\partial \beta^2}\right)\right]\right\},\end{aligned} \tag{8}$$

and

$$\begin{aligned}\frac{\partial^2 \ln \tilde{L}(\gamma,\beta,\lambda_0)}{\partial \gamma \partial \beta} = & \sum_{i=1}^{m}\frac{1}{\tilde{h}_i}\frac{\partial \tilde{h}_i}{\partial \beta} - \sum_{i=1}^{m}\left\{[k(\tilde{r}_i+1)-1]\left[\frac{\gamma\tilde{h}_i^{\gamma-1}\frac{\partial \tilde{h}_i}{\partial \beta}\ln\tilde{h}_i + \tilde{h}_i^{\gamma}\frac{1}{\tilde{h}_i}\frac{\partial \tilde{h}_i}{\partial \beta}}{\left(1-\tilde{h}_i^{\gamma}\right)}\right.\right. \\ & \left.\left. + \frac{\tilde{h}_i^{\gamma}\ln\tilde{h}_i\cdot\gamma\cdot\tilde{h}_i^{\gamma-1}\frac{\partial \tilde{h}_i}{\partial \beta}}{\left(1-\tilde{h}_i^{\gamma}\right)^2}\right]\right\},\end{aligned} \tag{9}$$

where $\frac{\partial \tilde{h}_i}{\partial \beta}$ as the above definition and

$$\frac{\partial^2 \tilde{h}_i}{\partial \beta^2} = \left(\ln\left(\frac{x_i}{\lambda_0}\right)\right)^2 \left(\frac{x_i}{\lambda_0}\right)^\beta \exp\left(-\left(\frac{x_i}{\lambda_0}\right)^\beta\right)\left[1-\left(\frac{x_i}{\lambda_0}\right)^\beta\right].$$

Under some mild regularity conditions (see Theorem 5.2.2 of Sen and Singer [7]), $(\hat{\gamma}, \hat{\beta})$ is asymptotically bivariately normal distribution with mean (γ, β) and covariance matrix $I^{-1}(\gamma, \beta)$, i.e., $(\hat{\gamma}, \hat{\beta}) \xrightarrow{D} N((\gamma, \beta),\ I^{-1}(\gamma, \beta))$ and the estimated asymptotic covariance matrix of $(\hat{\gamma}, \hat{\beta})$ is

$$I^{-1}(\hat{\gamma}, \hat{\beta}) = \left.\begin{bmatrix} E\left(-\frac{\partial^2 \ln \widetilde{L}(\gamma,\beta,\lambda_0)}{\partial \gamma^2}\right) & E\left(-\frac{\partial^2 \ln \widetilde{L}(\gamma,\beta,\lambda_0)}{\partial \gamma\, \partial \beta}\right) \\ E\left(-\frac{\partial^2 \ln \widetilde{L}(\gamma,\beta,\lambda_0)}{\partial \gamma\, \partial \beta}\right) & E\left(-\frac{\partial^2 \ln \widetilde{L}(\gamma,\beta,\lambda_0)}{\partial \beta^2}\right) \end{bmatrix}^{-1}\right|_{(\gamma,\beta)=(\hat{\gamma},\hat{\beta})}, \quad (10)$$

where $I^{-1}\left(\hat{\gamma}, \hat{\beta}\right)$ can be calculated by numerical integration with Compaq Visual Fortran version 6.6 [6] and IMSL subroutines QDAGI and QDAGS.

By the p-order quantile $x_p = F^{-1}(p) = h_1(\gamma, \beta) = \lambda_0\left[-\ln\left(1-p^{1/\gamma}\right)\right]^{1/\beta}$, $0 < p < 1$, and the delta method (see Theorem 5.5.28 of Casella and Berger [8]), we have the p-order quantile estimator $h_1\left(\hat{\gamma}, \hat{\beta}\right) \xrightarrow{D} N(h_1(\gamma,\beta), AVar(h_1(\gamma,\beta)))$, and the estimated asymptotic variance of the p-order quantile estimator $h_1\left(\hat{\gamma}, \hat{\beta}\right)$ is

$$AVar\left(h_1\left(\hat{\gamma}, \hat{\beta}\right)\right) = H_1\Big|_{(\gamma,\beta)=\left(\hat{\gamma},\hat{\beta}\right)} \cdot I^{-1}\left(\hat{\gamma}, \hat{\beta}\right) \cdot H_1^T\Big|_{(\gamma,\beta)=\left(\hat{\gamma},\hat{\beta}\right)}, \quad (11)$$

where

$$H_1 = \left(\frac{\partial h_1(\gamma,\beta)}{\partial \gamma}, \frac{\partial h_1(\gamma,\beta)}{\partial \beta}\right),$$

$$\frac{\partial h_1(\gamma,\beta)}{\partial \gamma} = \frac{\lambda_o}{\beta}\left[-\ln\left(1-p^{1/\gamma}\right)\right]^{\frac{1-\beta}{\beta}}\left[-(1-p^{1/\gamma})\right]\left(-p^{1/\gamma}\ln p\right)\left(-\gamma^{-2}\right),$$

$$\frac{\partial h_1(\gamma,\beta)}{\partial \beta} = \lambda_o\left[-\ln\left(1-p^{1/\gamma}\right)\right]^{1/\beta}\ln\left[-\ln\left(1-p^{1/\gamma}\right)\right]\left(-\beta^{-2}\right)$$

and by (10), $I^{-1}(\hat{\gamma}, \hat{\beta})$ can be calculated by numerical integration with Compaq Visual Fortran version 6.6 [6] and IMSL subroutines QDAGI and QDAGS.

By the survival function $S(x) = h_2(\gamma,\beta) = 1 - \left[1-\exp\left(-(x/\lambda_0)^\beta\right)\right]^\gamma$ and the delta method (see Theorem 5.5.28 of Casella and Berger [8]), we have survival function estimator $h_2\left(\hat{\gamma}, \hat{\beta}\right) \xrightarrow{D} N\left(h_2(\gamma,\beta), AVar\left(h_2(\gamma,\beta)\right)\right)$, and the estimated asymptotic variance of the survival function estimator $h_2\left(\hat{\gamma}, \hat{\beta}\right)$ is

$$AVar\left(h_2\left(\hat{\gamma}, \hat{\beta}\right)\right) = H_2\Big|_{(\gamma,\beta)=\left(\hat{\gamma},\hat{\beta}\right)} \cdot I^{-1}\left(\hat{\gamma}, \hat{\beta}\right) \cdot H_2^T\Big|_{(\gamma,\beta)=\left(\hat{\gamma},\hat{\beta}\right)}, \quad (12)$$

where

$$H_2 = \left(\frac{\partial h_2(\gamma,\beta)}{\partial \gamma}, \frac{\partial h_2(\gamma,\beta)}{\partial \beta}\right),$$

$$\frac{\partial h_2(\gamma,\beta)}{\partial \gamma} = -\left[1-\exp\left(-\left(\frac{x}{\lambda_o}\right)^\beta\right)\right]^\gamma \ln\left[1-\exp\left(-\left(\frac{x}{\lambda_o}\right)^\beta\right)\right],$$

$$\frac{\partial h_2(\gamma,\beta)}{\partial \beta} = -\gamma\left[1-\exp\left(-(x/\lambda_0)^\beta\right)\right]^{\gamma-1}(x/\lambda_0)^\beta \ln(x/\lambda_0)\exp\left(-(x/\lambda_0)^\beta\right)$$

and by (10), $I^{-1}(\hat{\gamma},\hat{\beta})$ can be calculated by numerical integration with Compaq Visual Fortran version 6.6 [6] and IMSL subroutines QDAGI and QDAGS.

4. **Planning of Life Test with Cost Constraint.** We assume that the proportion to be removed at the time of the ith failure, q_i, is pre-determined ($0 \le q_i < 1$) and the number of removed groups can be computed as $\tilde{r}_i = nq_i$, $i = 1, 2, \cdots, m$. Thus, a progressive first-failure censored variables sampling can be described by the number of groups n, the number of test units k in each group, and the proportions to be removed $q_1, q_2, \cdots, q_m$. The fixed cost has the installation cost C_a, the variable costs have the sample cost C_s, the total operation cost C_o and the budget of a life test C_T. Therefore, the total expected cost of the experiment is

$$TC(n,k) = C_a + nkC_s + E(X_m)C_o \le C_T, \tag{13}$$

where $N = nk$, $F(x) = \left[1 - \exp\left(-(x/\lambda)^{\beta}\right)\right]^{\gamma}$, $E(X_m)$ is the expected duration of the life test,

$$E(X_m) = \int_0^{\infty}\left\{\left(\prod_{i=1}^{m}\tilde{u}_i\right)\cdot\sum_{i=1}^{m}\left[\tilde{u}_i^{-1}\left[\prod_{\substack{j=1\\j\ne i}}^{m}(\tilde{u}_j-\tilde{u}_i)^{-1}\right][1-F(x)]^{\tilde{u}_i}\right]\right\}dx,$$

$$\begin{cases}\tilde{u}_i = N - i + 1 + \sum\limits_{j=i}^{m}\left(k\left(\tilde{r}_j+1\right)-1\right)\\ \tilde{u}_j = N - j + 1 + \sum\limits_{l=j}^{m}\left(k\left(\tilde{r}_l+1\right)-1\right)\end{cases}.$$

We investigate three selection criteria which enable one to choose the optimal value of (n,k), as follows: **(i)** Minimize the determinant of the estimated asymptotic variance-covariance matrix of maximum likelihood estimators, and thus, the first criterion function can be constructed by (10) as $G_1(n,k) = \det\left(I^{-1}\left(\hat{\gamma},\hat{\beta}\right)\right)$; **(ii)** Minimize the estimated asymptotic variance of p-order quantile estimator, and thus, the second criterion function can be constructed by (11) as $G_2(n,k) = AVar\left(h_1\left(\hat{\gamma},\hat{\beta}\right)\right)$; **(iii)** Minimize the estimated asymptotic variance of survival function estimator, and thus, the third criterion function can be constructed by (12) as $G_3(n,k) = AVar\left(h_2\left(\hat{\gamma},\hat{\beta}\right)\right)$.

The optimal design of sampling plan can be expressed with the criterion functions $G_i(n,k)$, $i = 1, 2, 3$, as follows respectively:

$$\text{Min } G_i(n,k),\ \exists i \in \{1,2,3\}$$
$$\text{Subject to } TC(n,k) = C_a + nkC_s + E(X_m)C_o \le C_T,\ n,k \in Z^+. \tag{14}$$

The progressive first-failure censored variables sampling plan can be described by n, k and $(q_1, q_2, \cdots, q_m)$. We may first fix the degree of censoring q ($0 < q < 1$) and then select the values of $q_1, q_2, \cdots, q_j, q_{j'}, \cdots, q_m$ such that $q_1 + q_2 + \cdots + q_j + q_{j'} + \cdots + q_m = q$ (see Balasooriya et al. [9]). Let m_1 denote the total number of selected $q_{i'}$s, the smallest number of groups to get m_1 observed failures has to satisfy $n \ge m_1/(1-q)$. Let m_2 be the smallest number of removed groups, the smallest number of groups to get m_2 removed groups has to satisfy $n \ge m_2/q$. So we have $n \ge \max\{m_1/(1-q), m_2/q\}$. By (14), we have $C_a + nkC_s + E(X_m)C_o \le C_T$. Because of the expected duration of the life test $E(X_m) > 0$, $C_a + nkC_s \leqq C_T$, hence $k \le (C_T - C_a)/(nC_s)$. n has the lower bound $n' = \max([m_1/(1-q)], [m_2/q])$, where the symbol $[x]$ denotes the largest integer which is $\le x$. So we can obtain the upper bound of k is $k' = [(C_T - C_a)/(n'C_s)]$.

Finally, the optimal solution of (14) can be obtained by an enumeration method. The algorithm included two parts (a) and (b), and the algorithm is stated as follows.

(a) The estimation of the parameters γ, β and λ:

Step a-1: Input n, k, m, $\{x_i, i = 1, \cdots, m\}$ and $(\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_m)$, where n is the number of groups, k is the number of test units in each group, m is the number of failures observed before termination, $\{x_i, i = 1, \cdots, m\}$ is the progressive first-failure censored order statistics data, and $(\tilde{r}_1, \tilde{r}_2, \cdots, \tilde{r}_m)$ is the censoring scheme. **Step a-2:** The maximum likelihood estimators $\hat{\gamma}$, $\hat{\beta}$ and $\hat{\lambda}$ are derived by solving the equations $\frac{\partial \ln \tilde{L}(\gamma,\beta,\lambda)}{\partial \gamma} = 0$, $\frac{\partial \ln \tilde{L}(\gamma,\beta,\lambda)}{\partial \beta} = 0$ and $\frac{\partial \ln \tilde{L}(\gamma,\beta,\lambda)}{\partial \lambda} = 0$ with Compaq Visual Fortran version 6.6 [6] and IMSL subroutine NEQNF.

(b) The solution (n, k) of the nonlinear programming with (14):

Step b-1: Input the values of cost parameters (C_T, C_o, C_a, C_s), the specified proportions of removals $(q_1, q_2, \cdots, q_j, q_{j'}, \cdots, q_m)$, the smallest number of observed failures m_1, the smallest number of removed groups m_2, and the maximum likelihood estimators $(\hat{\gamma}, \hat{\beta}, \hat{\lambda})$, where C_T is the budget of a life test, C_a is installation cost, C_o is the operation cost for each time unit, and C_s is sample cost. **Step b-2:** Calculate the lower bound of n and the upper bound of k. The lower bound of n is $n' = \max\left([m_1/(1-q)], [m_2/q]\right)$, where $q = q_1 + q_2 + \cdots + q_j + q_{j'} + \cdots + q_m$ is the degree of censoring, and the symbol $[x]$ denotes the largest integer which is $\leq x$. The upper bound of k is $k' = [(C_T - C_a)/(n'C_s)]$ **Step b-3:** Set $k = 1$. **Step b-4:** Set $n_k = n' + 1$. **Step b-5:** Compute $G(n_k^*, k) = \min_A \{G(n_k - 1, k), G(n_k, k)\}$, $A = \{TC(n_k - 1, k) \leq C_T\} \cup \{TC(n_k, k) \leq C_T\}$, and $\lambda_0 = \hat{\lambda}$. **Step b-6:** Set $k = k + 1$, if $k \leq k'$, go to **Step b-4**; else go to **Step b-7**. **Step b-7:** Calculate the minimum value of the criterion function $G(n, k)$. That is, $G\left(\tilde{n}, \tilde{k}\right) = \min_{\{1 \leq k \leq k'\}} \{G(n_k^*, k)\}$. **Step b-8:** The optimal design $\left(\tilde{n}, \tilde{k}\right)$ is obtained.

5. **Numerical Example.** Nelson [10] presents the results of a life-test experiment in which specimens of a type of electrical insulating fluid were subject to a constant voltage stress. The length of time until each specimen failed (or brokedown) was observed. The vector of observed failure times and the progressive first-failure censoring scheme are given as follow: $\{x_i, i = 1, \cdots, 8\} = (0.19, 0.78, 0.96, 1.31, 2.78, 4.85, 6.50, 7.35)$, $\tilde{r} = (0, 0, 3, 0, 3, 0, 0, 5)$, $n = 19$, $m = 8$ and $k = 1$. Then, the optimal solution of (14) is as follows.

(i) The estimation of the parameters γ, β and λ.

By solving the equations $\frac{\partial \ln \tilde{L}(\gamma,\beta,\lambda)}{\partial \gamma} = 0$, $\frac{\partial \ln \tilde{L}(\gamma,\beta,\lambda)}{\partial \beta} = 0$ and $\frac{\partial \ln \tilde{L}(\gamma,\beta,\lambda)}{\partial \lambda} = 0$, we can attain the maximum likelihood estimates are $\hat{\gamma} = 0.7708132$, $\hat{\beta} = 1.223805$ and $\hat{\lambda} = 8.820102$. Here, we fixed $\lambda_0 = 8.820102$.

(ii) The solution (n, k) of the nonlinear programming with (14).

Input the values of cost parameters $C_T = 120$, $C_o = 10$, $C_a = 40$, $C_s = 1$, the specified proportions of removals $q_3 = 3/19$, $q_5 = 3/19$, $q_8 = 5/19$, the smallest number of observed failures $m_1 = 3$, the smallest number of removed groups $m_2 = [(3/19 + 3/19 + 5/19)/(3/19)] = 3$. The lower bound of n is $n' = \max([3/(1 - 11/19)], [3/(11/19)]) = 7$. The upper bound of k is $k' = [(C_T - C_a)/(n'C_s)] = [(120 - 40)/(7 \times 1)] = 11$. Table 1 shows the optimal solution of (n, k) under the three criterion functions with $p = 0.5$. The optimal solution of (n, k) is $(7, 11)$ under the three criterion functions. The optimal solution of the first criterion function $G_1(n, k)$ is 2.893×10^{-4}, and it is best of the three criterion functions.

6. **Sensitivity Analysis.** The sensitivity study of the optimal solution to change in the values of the different parameters is an important issue to the planning of lifetime test. In numerical example, we attained the maximum likelihood estimates $\hat{\gamma} = 0.7708132$,

Table 1. The optimal solution of (n, k) under the three criterion functions

The criterion function	The optimal solution $\tilde{n}$ of n	The optimal solution $\tilde{k}$ of k	The optimal solution of the criterion function
$G_1(n, k)$	7	11	2.893×10^{-4}
$G_2(n, k)$	7	11	0.35002
$G_3(n, k)$	7	11	1452×10^3

$\hat{\beta} = 1.223805$ and $\hat{\lambda} = 8.820102$, respectively, and fixed the values of cost parameters $C_T = 120$, $C_o = 10$, $C_a = 40$, $C_s = 1$. Using the same estimates of distribution parameters $\hat{\gamma} = 0.7708132$, $\hat{\beta} = 1.223805$ and $\lambda_0 = 8.820102$, the influence of cost parameters (C_T, C_o, C_a, C_s) on n, k and $G_i(n, k)$, $i = 1, 2, 3$. We investigated the effects of the parameters of experimental cost (C_T, C_o, C_a, C_s) on n, k and $G_i(n, k)$, $i = 1, 2, 3$ as follows: **(i)** $\tilde{n}$ is constant, $\tilde{k}$ is increasing and $G_i\left(\tilde{n}, \tilde{k}\right)$ is decreasing, as C_T increases. **(ii)** $\tilde{n}$ is constant, $\tilde{k}$ is decreasing and $G_i\left(\tilde{n}, \tilde{k}\right)$ is increasing, as C_s increases. **(iii)** $\tilde{n}$ is constant, $\tilde{k}$ is constant and $G_i\left(\tilde{n}, \tilde{k}\right)$ is constant, as C_o increases. **(iv)** $\tilde{n}$ is constant, $\tilde{k}$ is decreasing and $G_i\left(\tilde{n}, \tilde{k}\right)$ is increasing, as C_a increases. **(v)** $\tilde{n}$ is constant, $\tilde{k}$ and $G_i\left(\tilde{n}, \tilde{k}\right)$ are negative correlation, as (C_T, C_a, C_s) increase. **(vi)** $G_i\left(\tilde{n}, \tilde{k}\right)$ is increasing, as (C_a, C_s) increase. By **(i)-(vi)**, $G_i\left(\tilde{n}, \tilde{k}\right)$ is sensitive to changes in C_T, C_a and C_s, but $G_i\left(\tilde{n}, \tilde{k}\right)$ is insensitive to changes in C_o. We can find that the choice of the values of C_T, C_a and C_s is important to $G_i\left(\tilde{n}, \tilde{k}\right)$.

7. **Conclusions.** In this paper, we use the method of D-optimality, p-order quantile estimator, and survival function to determine the optimum solutions of the number of testing group (n) and the number of test in group (k) under the progressive first-failure censoring plan for the exponentiated Weibull products with restrictions relating to the cost of the life testing, respectively. The decision problem of obtaining the number of testing group (n) and the number of test in group (k) under restricted budget of experiment is important for experimenters. The simulation results indicate that the choice of the values of cost parameters is important. The proposed approach not only helps us to make decision in product management, but also provides the most efficient use of experimenter's resources. In future research on this optimal decision problem, it would be interesting to deal with the objective function and the total cost of constraint based on progressive type I group censoring sample.

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