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A NOVEL ALGORITHM FOR SOLVING THE SHORTEST PATH PROBLEM UNDER UNCERTAIN ENVIRONMENT

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Abstract. *Identifying the shortest path is a subject of great practical importance to many researchers. In the past decade, many studies have investigated the shortest path problem with crisp link cost. However, in many transportation networks, the link cost usually varies from day to day due to the variability in the traffic demand. In this paper, we develop a Dijkstra algorithm to deal with the shortest path problem under uncertain environment. To characterize the uncertainty of link cost, various fuzzy numbers are implemented to achieve this objective. A unified distance function is defined to convert different types of fuzzy numbers into crisp numbers. Then Dijkstra algorithm is used to find the shortest path in the network. A numerical example is used to illustrate the efficiency of the proposed method.*
Keywords: Shortest path, Fuzzy number, Dijkstra algorithm, Uncertainty

1. **Introduction.** Finding the shortest path between two nodes is a subject of great practical importance to planners and engineers involved in many network optimization problems. Since it is widely used in many applications, e.g., road navigation [1], network design [2], this problem has attracted much attention from many researchers. A number of studies have been performed to reduce the computational time for finding the shortest path between two nodes and many algorithms have been developed to find the shortest path in large scale networks. For example, Bertsekas [3] proposed a simple but efficient label correcting algorithm for solving the shortest path problem, in which he marked each node using different labels and scanned the candidate nodes with an ascending labels. Iori et al. [4] presented a label setting algorithm for solving multi-objective shortest path problems by aggregating the ordering of the labels. Among these methods, Dijkstra algorithm [5] is one of the most commonly used algorithms to identify the minimum cost path in many network optimization problems. For example, Yin and Wang [6] developed an improved Dijkstra algorithm, in which they considered different types of weights to reduce the computational time.

However, in the above studies, the link cost is treated as static, whereas, in many realistic scenarios, the link cost usually varies with time. For example, in many transportation networks, due to the variability in the traffic demand, the travel time related to each link varies provided that the link cost is a function of link flow [7]. Additionally, in the wireless network, data packet is forwarded along the minimum cost path. However, due to the uncertainty in demand, different degrees of congestion arise in the network, which in turn affect the packet transfer time [8]. Among the literature, many studies have used fuzzy numbers to characterize the uncertainty associated with each link [9, 10]. For example, Okada and Soper [11] employed fuzzy number to represent the arc length and defined a fuzzy min operator to find out the nondominated path in the network. Kung and Chuang

[12] considered the fuzzy shortest path problem and defined a similarity metric to quantify the similarity among different fuzzy shortest paths.

However, fuzzy numbers have also been criticized due to several crucial deficiencies. First of all, by using fuzzy numbers, the shortest path found by the algorithm is also a fuzzy number. Due to the uncertainty in arc length, several candidate shortest paths are available in the final results. However, how to order the fuzzy numbers is still an open issue. Secondly, if the link cost is represented by several different types of fuzzy numbers, e.g., normal fuzzy numbers or triangular fuzzy numbers, many arithmetic operations, such as, fuzzy addition, fuzzy subtraction, and fuzzy multiplications, become infeasible because they are designed only for the fuzzy numbers in the same form. For example, as in [9], since their method is only able to handle the trapezoidal fuzzy numbers, the method cannot handle various types of fuzzy arc lengths. As a result, we cannot compare the proposed method in this paper with it.

Consider the various deficiencies existing in the current methods, we are motivated to develop a new algorithm to overcome these drawbacks to address the shortest path problem in the uncertain environment. Specifically, in the proposed method, fuzzy numbers are still used to characterize the uncertainty of arc length. However, instead of applying fuzzy operators, we propose a unified distance function to convert fuzzy numbers into crisp numbers, by which we avoid the deficiencies of the aforementioned methods. More importantly, we do not need any arithmetic operation among the fuzzy numbers. Last but not the least, since the proposed method does not need any arithmetic operations among fuzzy numbers, it reduces the computational time extensively, which makes it possible to be implemented into large networks.

The remainder of this paper is structured as follows. The remainder of this paper is organized as follows. In Section 2, the fuzzy numbers are briefly introduced. In Section 3, we present the proposed method. In Section 4, we present the experimental results and analysis. Finally, we give our conclusions in Section 5.

2. **Preliminaries.** In 1965, the notion of fuzzy sets was first introduced by Zadeh [13], providing a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criterion of class membership. Fuzzy set is provided by the definitions below.

Definition 2.1. *A fuzzy set $\widetilde{A}$ defined on a universe X may be given as:*

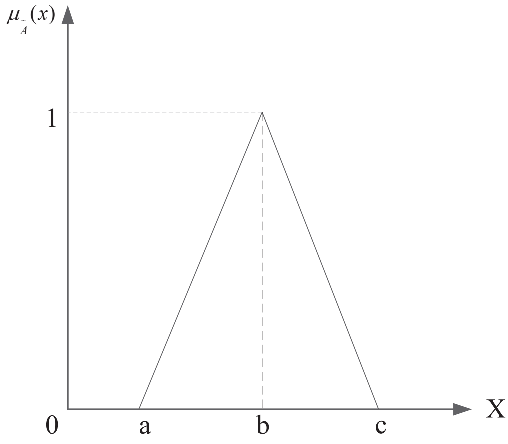
$$\widetilde{A} := \left\{ \left\langle x, \mu_{\widetilde{A}}(x) \right\rangle | x \in X \right\},$$

where $\mu_A : X \to [0,1]$ is the membership function A. The membership value $\mu_A(x)$ describes the degree of belongingness of $x \in X$ in A.

Definition 2.2. *A triangular fuzzy number $\widetilde{A}$ can be defined by a triplet (a,b,c), where the membership can be determined as follows*

$$\mu_{\widetilde{A}}(x) := \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \le x \le b, \\ \frac{c-x}{c-b}, & b \le x \le c, \\ 0, & x > c. \end{cases} \tag{1}$$

A triangular fuzzy number $\widetilde{A} = (a,b,c)$ is shown diagrammatically in Figure 1(a).



(a) A triangular fuzzy number (b) A trapezoidal fuzzy number

Figure 1. Two kinds of fuzzy numbers

Definition 2.3. *A trapezoidal fuzzy number* $\widetilde{A}$ *can be defined as* $\widetilde{A} = (a_1, a_2, a_3, a_4)$*, where the membership can be determined as follows*

$$\mu_{\widetilde{A}}(x) := \begin{cases} 0, & x < a_1, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ 1, & a_2 \leq x \leq a_3, \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4, \\ 0, & x > c. \end{cases} \tag{2}$$

A trapezoidal fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4)$ is shown diagrammatically in Figure 1(b).

Definition 2.4. *Iff* $L(x) = R(x) = e^{-x^2}$*, with* $x \in \mathbb{R}$*, then* x *is a normal fuzzy number that is shown by* (m, σ) *and the corresponding membership function is defined to be:*

$$\mu_{\widetilde{A}}(x) := e^{-\left(\frac{x-m}{\sigma}\right)^2} \quad x \in \mathbb{R},$$

where m *is the mean and* σ *is the standard deviation. A normal fuzzy number is shown in Figure 2.*

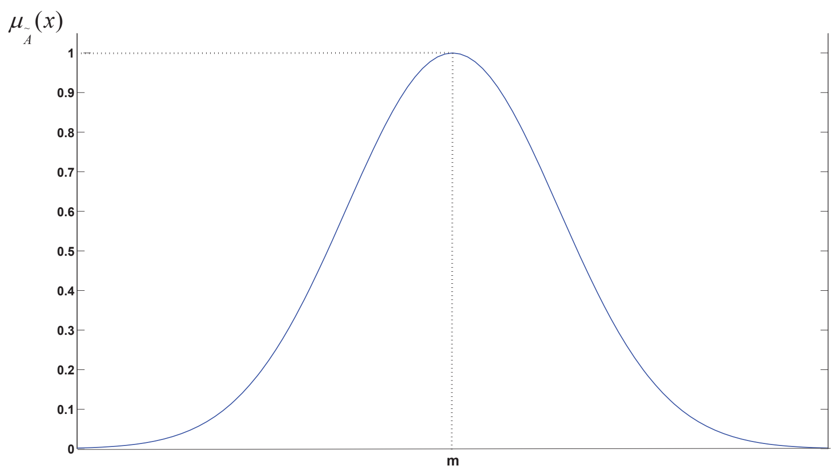


Figure 2. A normal fuzzy number

Definition 2.5. *α-cuts for trapezoidal fuzzy numbers: Suppose $\widetilde{a} = (a_1, a_2, a_3, a_4)$. An α-cuts for $\widetilde{a}$, $\widetilde{a}_\alpha$, is calculated as:*

$$\left.\begin{array}{l} \alpha := \frac{x-a_1}{a_2-a_1} \\ \alpha := \frac{a_4-x}{a_4-a_3} \end{array}\right\} \Rightarrow \widetilde{a}_\alpha := \left\{\begin{array}{l} \widetilde{a}_\alpha^L := x := (a_2 - a_1)\,\alpha + a_1, \\ \\ \widetilde{a}_\alpha^R := x := a_4 - (a_4 - a_3)\,\alpha, \end{array}\right. \quad 0 < \alpha \leq 1, \tag{3}$$

where $\widetilde{a}_\alpha \left[\widetilde{a}_\alpha^L, \widetilde{a}_\alpha^R\right]$ is the corresponding α-cut, $\widetilde{a}_\alpha^L$ denotes the lower bound of α-cuts for trapezoidal fuzzy numbers, and $\widetilde{a}_\alpha^R$ represents the upper bound for α-cuts for trapezoidal fuzzy numbers. The cuts for triangular fuzzy numbers can be obtained by using the above equations considering $a_2 = a_3$.

Definition 2.6. *α-cuts for normal fuzzy numbers: Assume $\widetilde{a} = (m, \sigma)$ is a normal fuzzy number, then $\widetilde{a}_\alpha$ is computed as:*

$$\left.\begin{array}{l} \alpha = e^{-\left(\frac{x-m}{\sigma}\right)^2} \\ \alpha = e^{-\left(\frac{x-m}{\sigma}\right)^2} \end{array}\right\} \Rightarrow \widetilde{a}_\alpha = \left\{\begin{array}{l} \widetilde{a}_\alpha^L = x = m - \sigma\sqrt{-\ln(\alpha)}, \\ \\ \widetilde{a}_\alpha^R = x = m + \sigma\sqrt{-\ln(\alpha)}, \end{array}\right. \quad 0 < \alpha \leq 1. \tag{4}$$

3. **Proposed Method.** In this section, a function is defined to convert the fuzzy numbers into crisp numbers. Then we develop a Dijkstra algorithm to find the shortest path in uncertain environment.

3.1. **Distance of fuzzy numbers.** Consider the fuzzy *min* operator between any two fuzzy numbers. Suppose we have two trapezoidal fuzzy numbers as below:

$$\widetilde{a} = (a_1, a_2, a_3, a_4), \quad \widetilde{b} = (b_1, b_2, b_3, b_4)$$

According to the fuzzy operator defined in [14], we have:

$$Min\ value\ \left(\widetilde{a}, \widetilde{b}\right) = (\min(a_1, b_1), \min(a_2, b_2), \min(a_3, b_3), \min(a_4, b_4))$$

However, in many cases, the above fuzzy *min* operator might result in a fuzzy number distinct from both of them. For instance, given two fuzzy numbers $\widetilde{a} = (10, 20, 25, 31)$ and $\widetilde{b} = (12, 18, 29, 31)$, after applying the fuzzy *min* operator, we have $Min\ value\ \left(\widetilde{a}, \widetilde{b}\right) = (10, 18, 25, 31)$. As observed, the result is different from both $\widetilde{a}$ and $\widetilde{b}$. To get rid of the drawback of this method, we define a distance function [15].

Definition 3.1. *$D_{p,q}$-distance: Given two fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$, then the $D_{p,q}$-distance between them can be defined as follows:*

$$D_{p,q}\left(\widetilde{a}, \widetilde{b}\right) := \left\{\begin{array}{ll} \left[(1-q)\int_0^1 \left|a_\alpha^- - b_\alpha^-\right|^p d\alpha + q\int_0^1 \left|a_\alpha^+ - b_\alpha^+\right|^p d\alpha\right]^{\frac{1}{p}}, & p < \infty \\ (1-q)\sup\limits_{0<\alpha\leq 1} \left|a_\alpha^- - b_\alpha^-\right| + q\inf\limits_{0<\alpha\leq 1} \left|a_\alpha^+ - b_\alpha^+\right|, & p = \infty \end{array}\right. \tag{5}$$

where the first parameter p denotes the priority weight attributed to the end points of the support; for instance, the a_α^+ and a_α^- of the fuzzy numbers. Iff the expert has no preference, $D_{p,\frac{1}{2}}$ is recommended. The second parameter q determines the analytical properties of $D_{p,q}$.

For two fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$, the $D_{p,q}$ can be approximately proportional to:

$$D_{p,q}\left(\widetilde{a}, \widetilde{b}\right) := \left[(1-q)\sum_{i=1}^{n} \left|a_\alpha^- - b_\alpha^-\right|^p + q\sum_{i=1}^{n} \left|a_\alpha^+ - b_\alpha^+\right|^p\right]^{\frac{1}{p}}. \tag{6}$$

Iff $q = 1/2$, $p = 2$, then (6) can be turned into:

$$D_{p,q}\left(\widetilde{a}, \widetilde{b}\right) := \sqrt{\frac{1}{2}\sum_{i=1}^{n} \left|a_\alpha^- - b_\alpha^-\right|^2 + \frac{1}{2}\sum_{i=1}^{n} \left|a_\alpha^+ - b_\alpha^+\right|^2}. \tag{7}$$

Herein, to express the fuzzy numbers using crisp numbers, we use the distance from the fuzzy numbers to $\widetilde{MV} = (0, 0, \cdots, 0)$ as the true value of fuzzy number. Since the lower bound and upper bound α-cuts are 0 and 1, we divide the α-interval into n subintervals by letting $\alpha_0 = 0$, $\alpha_i = \alpha_{i-1} + \Delta\alpha_i$, where $\Delta\alpha_i = \frac{1}{n}$, $i = 1, 2 \cdots, n$. In the case of normal fuzzy numbers, it is not reasonable to set α equal to 0. To deal with normal fuzzy numbers, we let $\alpha \in (0, 1]$. Based on Equations (3) and (4), we compute the α-cuts for fuzzy numbers. Afterwards, by applying the operations defined in Equation (7), the fuzzy numbers can be converted into crisp numbers.

3.2. **Fuzzy Dijkstra algorithm.** Since we are capable of transforming all the fuzzy numbers into crisp numbers, by implementing the most commonly used Dijkstra algorithm, we are able to find the shortest path in the network. In general, Dijkstra algorithm assigns initial values to each node: set it to zero for the starting node and to infinity for all the other nodes. Next, the algorithm identifies a node u from the unvisited node set Q that has the minimum distance to the source node. Afterwards, node u is removed from the unvisited set Q. For the current node u, it considers all of its unvisited neighbors, calculates their tentative distances, and compares the newly calculated tentative distance to the current assigned value and chooses the smaller one. The process continues until the candidate node is the sink node t. The flow chart of the proposed method is shown in Algorithm 1.

Algorithm 1: Dijkstra Algorithm for Fuzzy Shortest Path Problem

Data: $G = (L, V, E, s, t)$, where L is an adjacency matrix of graph G, V is the set of nodes, E is the set of edges, s is the starting node, and t is the ending node.
Result: The shortest path from s to t.

$S \leftarrow \emptyset$
$Q \leftarrow V$
$last = s$
$dist[last] = 0$
while $last \neq t$ **do**
    Converting fuzzy numbers into crisp numbers using Equation (7);
    u $\leftarrow$ vertex in Q with min dist[u];
    $last \leftarrow u$;
    $S = S \cup u$
    for *all* $u \in$ *neighbors[u] and* $u \notin S$ **do**
        if $dist[v] > dist[u] + d[u, v]$ **then**
            $dist[v] = dist[u] + d[u, v]$

4. **Numerical Example.** In this section, a numerical example is used to demonstrate the efficiency of the proposed method. Consider the network shown in Figure 3, there are 11 nodes and 25 edges in the network. Each edge has a different edge length from each other, which are represented by two different types of fuzzy numbers (normal fuzzy numbers and triangular fuzzy numbers). The specific characteristic of each link is shown in Table 1.

Now, the shortest path between nodes 1 and 11 needs to be found. After implementing the developed fuzzy Dijkstra algorithm, we find the shortest path from node 1 to node 11: $1 \rightarrow 3 \rightarrow 8 \rightarrow 7 \rightarrow 11$. The result is also compared with the result in [16]. In [16], they developed a genetic algorithm to find the shortest path by four steps: path representation, crossover, mutation, and strategy analysis. However, in this algorithm, a lot of generated

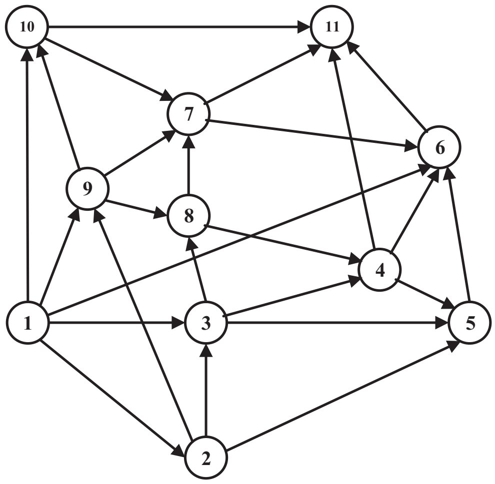


Figure 3. A network with 11 nodes

Table 1. The fuzzy edge lengths for the network shown in Figure 3

Arc	Fuzzy number	Arc	Fuzzy number	Arc	Fuzzy number
(1,2)	(800,820,840)	(3,5)	(730,748,870)	(8,4)	(710,730,833)
(1,3)	(35,11)	(3,8)	(42,14)	(8,7)	(230,242,355)
(1,6)	(650,677,783)	(4,5)	(190,199,310)	(9,7)	(120,130,250)
(1,9)	(290,300,350)	(4,6)	(310,340,360)	(9,8)	(13,4)
(1,10)	(420,450,570)	(4,11)	(71,23)	(9,10)	(23,7)
(2,3)	(180,186,293)	(5,6)	(610,660,790)	(10,7)	(330,342,450)
(2,5)	(495,510,625)	(6,11)	(23,7)	(10,11)	(125,41)
(2,9)	(90,30)	(7,6)	(390,410,540)	(3,4)	(650,667,983)
(7,11)	(45,15)				

populations have been discarded because they cannot constitute a feasible path. Hence, it takes a considerable amount of time to generate feasible solutions. In addition, since crossover and mutation might result in infeasible solutions, additional operations are required to adjust the infeasible solutions, which makes their method limited to networks with small size. The toy problem with only 4 nodes is solid proof of this deficiency. However, in the proposed method, we do not need any additional operations to process the found path. Moreover, the proposed method can account for the fuzzy numbers in different types. As shown in Table 1, the arc length is represented by different fuzzy numbers, e.g., normal fuzzy number, triangular fuzzy numbers. In terms of accuracy, the proposed method obtains the same shortest path to connect node 1 with 11.

5. **Conclusion.** In this paper, we investigate the shortest path problem under uncertain environment, in which the link cost is represented by different types of fuzzy numbers. A distance function is defined to transform the fuzzy numbers into crisp numbers. Afterwards, Dijkstra algorithm is used to find the shortest path in the graph. The advantage of the proposed method is that it does not require any arithmetic operators among fuzzy numbers, which makes it capable of processing different types of fuzzy numbers. Future researches can be carried out in the following directions. First of all, in the traffic network, since the link cost is a function of link flow, how to relate the link cost uncertainty with the link flow is worthy to investigate. Secondly, when passengers choose the optimal path, they usually consider several criteria, e.g., travel time, cost, safety. Hence, it is necessary to develop an algorithm to address the multi-criteria network optimization in the presence of uncertainty.

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