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SOME RESULTS AND APPLICATIONS ON PGTOR ITERATIVE ALGORITHMS FOR SOLVING LINEAR SYSTEMS

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Abstract. *In this paper, we first present the generalized two-parameter overrelaxation (GTOR) iterative method for solving linear system $Ax = b$. Two kinds of preconditioners $\widetilde{S}$ are proposed. We set A and $\widetilde{S}$ to be the same 2×2 block structure. Next, we provide the convergence analysis. The results show that our preconditioners increase the convergence rate of the GTOR iterative methods. Finally, we give the iterative algorithms and present a numerical example to illustrate the theoretical results.*

Keywords: Preconditioned GAOR iterative method, Preconditioned GTOR iterative method, Irreducible, Central difference scheme

1. **Introduction.** In this paper, we consider the following linear system:

$$Ax = b, \tag{1}$$

where $A \in R^{n\times n}$ is a nonsingular matrix and $x, b \in R^n$.

To solve the linear system (1), the generalized accelerated overrelaxation (GAOR) iterative method has been used widely where A is split into $A = I - L - U$ (see [1, 2, 3, 4, 5]). As is well-known, all the elements in L and U are considered as a whole in the GAOR scheme. It does not seem reasonable since the case where the elements of A vary a lot in magnitude usually occurs in practice. Thus, we also consider the case that the lower part of A is split into two parts.

In this case, we assume that $A = I - C_{L1} - C_{L2} - C_U$ where I is the identity matrix, C_U is strictly upper triangular matrix, while C_{L1} and C_{L2} are strictly lower triangular matrices, and then the two-parameter overrelaxation (TOR) iterative method is proposed by [6, 7].

In order to accelerate the convergence rates of the GTOR iterative methods, an effective way is to transform the original system into the preconditioned form

$$\widetilde{A}x = \widetilde{b}, \tag{2}$$

where $\widetilde{A} = PA = \left(I + \widetilde{S}\right)A$ and $\widetilde{b} = Pb = \left(I + \widetilde{S}\right)b$.

Unlike the discussion in those papers, we set matrix A to be 2×2 block structure in this paper. If C_U is ordinary upper triangular matrix, in order to solve (1), using the thought of dividing matrix to blocks, we split A as

$$A = I - \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_1 & \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_2 & \mathbf{0} \end{pmatrix} - \begin{pmatrix} B_1 & -D \\ \mathbf{0} & B_2 \end{pmatrix},$$

where $C = C_1 + C_2 = (c_{ij})_{(n-p)\times p}$, $B_1 = \left(b_{ij}^{(1)}\right)_{p\times p}$, $B_2 = \left(b_{ij}^{(2)}\right)_{(n-p)\times(n-p)}$ and $D = (d_{ij})_{p\times(n-p)}$. Thus, we get a generalized TOR (GTOR) iterative method which can be defined by

$$x^{(k+1)} = \mathcal{T}_{\omega,\gamma}x^{(k)} + \frac{\omega+\gamma}{2}g, \tag{3}$$

where

$$\mathcal{T}_{\omega,\gamma} = \begin{pmatrix} I & \mathbf{0} \\ \frac{\gamma}{2}C_1 + \frac{\omega}{2}C_2 & I \end{pmatrix}^{-1} \left[\left(1 - \frac{\gamma}{2} - \frac{\omega}{2}\right) I + \frac{\omega}{2} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_1 & \mathbf{0} \end{pmatrix} \right.$$
$$\left. + \frac{\gamma}{2} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_2 & \mathbf{0} \end{pmatrix} + \frac{\gamma+\omega}{2} \begin{pmatrix} B_1 & -D \\ \mathbf{0} & B_2 \end{pmatrix} \right] \tag{4}$$
$$= \begin{pmatrix} \left(1 - \frac{\gamma}{2} - \frac{\omega}{2}\right) I + \frac{\gamma+\omega}{2}B_1 & -\frac{\gamma+\omega}{2}D \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{pmatrix} \tag{5}$$

with

$$\mathcal{T}_{21} = \frac{\gamma+\omega}{2}\left(\frac{\gamma}{2} - 1\right) C_1 - \frac{(\gamma+\omega)\gamma}{4}C_1B_1 + \frac{\gamma+\omega}{2}\left(\frac{\omega}{2} - 1\right)C_2 - \frac{(\gamma+\omega)\omega}{4}C_2B_1,$$
$$\mathcal{T}_{22} = \left(1 - \frac{\gamma}{2} - \frac{\omega}{2}\right) I + \frac{\gamma+\omega}{2}B_2 + \frac{(\gamma+\omega)\gamma}{4}C_1D + \frac{(\gamma+\omega)\omega}{4}C_2D,$$

and

$$g = \begin{pmatrix} I & \mathbf{0} \\ \frac{\gamma}{2}C_1 + \frac{\omega}{2}C_2 & I \end{pmatrix}^{-1} b = \begin{pmatrix} I & \mathbf{0} \\ -\frac{\gamma}{2}C_1 - \frac{\omega}{2}C_2 & I \end{pmatrix} b.$$

Set $\widetilde{S}$ to be a 2×2 block matrix [8]

$$\widetilde{S} = \begin{pmatrix} S & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{6}$$

where S is a $p\times p$ nonsingular matrix with $p < n$ and $P \in R^{n\times n}$. Obviously, $\widetilde{S}$ is with the same block form as matrix A.

Choosing different kinds of S, which is denoted by S_i, $(i = 1, 2, \ldots)$, we can express the coefficient matrix of (2) as

$$\widetilde{A_i} = I - \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_1 & \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_2 & \mathbf{0} \end{pmatrix} - \begin{pmatrix} [B_1 - S_i(I - B_1)] & v - (I + S_i)D \\ \mathbf{0} & B_2 \end{pmatrix}, \tag{7}$$
$$i = 1, 2, \ldots.$$

Then the preconditioned GTOR (PGTOR) iterative methods for solving (2) are defined as follows

$$x^{(k+1)} = \mathcal{T}_{\omega,\gamma}^{(i)}x^{(k)} + \frac{\omega+\gamma}{2}g, \quad i = 1, 2, \ldots, \tag{8}$$

where

$$\mathcal{T}_{\omega,\gamma}^{(i)} = \begin{pmatrix} I & \mathbf{0} \\ \frac{\gamma}{2}C_1 + \frac{\omega}{2}C_2 & I \end{pmatrix}^{-1} \left[\left(1 - \frac{\gamma}{2} - \frac{\omega}{2}\right) I \right.$$
$$+ \frac{\gamma+\omega}{2} \begin{pmatrix} B_1 - S_i(I - B_1) & -(I + S_i)D \\ \mathbf{0} & B_2 \end{pmatrix}$$
$$\left. + \frac{\omega}{2} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_1 & \mathbf{0} \end{pmatrix} + \frac{\gamma}{2} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_2 & \mathbf{0} \end{pmatrix} \right] \tag{9}$$

$$= \begin{pmatrix} \left(1-\frac{\gamma}{2}-\frac{\omega}{2}\right)I+\frac{\gamma+\omega}{2}[B_1-S_i(I-B_1)] & -\frac{\gamma+\omega}{2}(I+S_i)D \\ \mathcal{T}_{21}^{(i)} & \mathcal{T}_{22}^{(i)} \end{pmatrix} \tag{10}$$

with

$$\mathcal{T}_{21}^{(i)} = \frac{\gamma+\omega}{2}\left(\frac{\gamma}{2}-1\right)C_1 - \frac{(\gamma+\omega)\,\gamma}{4}C_1\left[B_1-S_i\left(I-B_1\right)\right] - \frac{(\gamma+\omega)\,\omega}{4}C_2\left[B_1-S_i\left(I-B_1\right)\right] + \frac{\gamma+\omega}{2}\left(\frac{\omega}{2}-1\right)C_2,$$

$$\mathcal{T}_{22}^{(i)} = \left(1-\frac{\gamma}{2}-\frac{\omega}{2}\right)I + \frac{(\gamma+\omega)\,\gamma}{4}C_1\left(I+S_i\right)D + \frac{(\gamma+\omega)\,\omega}{4}C_2\left(I+S_i\right)D + \frac{\gamma+\omega}{2}B_2.$$

This paper is organized as follows. In Section 2, we first present some results, then two kinds of preconditioners are proposed and the comparison conclusions are obtained, which are not only between the preconditioned and original methods but also between different preconditioned methods. The results show that our preconditioners increase the convergence rate of the GTOR iterative method. In Section 3, algorithms are presented. In Section 4, an example is proposed to show the effectiveness of the present algorithms. Finally, Section 5 concludes the paper.

2. Convergence Analysis and Comparisons.

For later reference, we need the following results.

Lemma 2.1. [9] *Let A be a nonnegative and irreducible $n \times n$ matrix. Then,*

(i) *A has a positive real eigenvalue equal to its spectral radius $\rho(A)$.*
(ii) *To the spectral radius $\rho(A)$, there corresponds an eigenvector $x > \mathbf{0}$ such that $Ax = \rho(A)x$.*

Lemma 2.2. [10] *Let A be a nonnegative and irreducible $n \times n$ matrix. If $\mathbf{0} \neq \alpha x \leq Ax \leq \beta x$, $\alpha x \neq Ax$, $Ax \neq \beta x$ for some nonnegative vector x, then $\alpha < \rho(A) < \beta$ and x is a positive vector.*

In (6), we take two kinds of S as follows.

$$S_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{b_{p1}}{\alpha} & 0 & \cdots & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{b_{p1}}{\alpha}+\beta & 0 & \cdots & 0 \end{pmatrix},$$

where b_{ij} are related to $b_{ij}^{(1)}$ in matrix B_1, and α, β are randomly chosen real parameters.

We have

$$B_1 - S_1(I-B_1) = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p-1,1} & b_{p-1,2} & \cdots & b_{p-1,p} \\ b_{p1} - \frac{b_{p1}}{\alpha}(1-b_{11}) & b_{p2}+\frac{b_{p1}}{\alpha}b_{12} & \cdots & b_{pp}+\frac{b_{p1}}{\alpha}b_{1p} \end{pmatrix},$$

$$B_1 - S_2(I-B_1) = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p-1,1} & b_{p-1,2} & \cdots & b_{p-1,p} \\ b_{p1} - \left(\frac{b_{p1}}{\alpha}+\beta\right)(1-b_{11}) & b_{p2}+\left(\frac{b_{p1}}{\alpha}+\beta\right)b_{12} & \cdots & b_{pp}+\left(\frac{b_{p1}}{\alpha}+\beta\right)b_{1p} \end{pmatrix}.$$

Now, we first discuss the convergence of the PGTOR methods and give comparisons between the GTOR and PGTOR methods.

Theorem 2.1. *Let $\mathcal{T}_{\omega,\gamma}$ and $\mathcal{T}_{\omega,\gamma}^{(1)}$ be the iteration matrices associated to the GTOR and PGTOR methods, respectively. If the matrix A in (1) is irreducible with $D \le \mathbf{0}$, $B_1 \ge \mathbf{0}$, $B_2 \ge \mathbf{0}$, $C_1 \le \mathbf{0}$, $C_2 \le \mathbf{0}$, $0 < \omega + \gamma \le 2$, $\omega > 0$, $\gamma > 0$, $b_{p1} > 0$, $\alpha > 0$, $\alpha > 1 - b_{11}$, then*

(i) *If $\rho(\mathcal{T}_{\omega,\gamma}) < 1$, then $\rho\left(\mathcal{T}_{\omega,\gamma}^{(1)}\right) < \rho(\mathcal{T}_{\omega,\gamma})$;*

(ii) *If $\rho(\mathcal{T}_{\omega,\gamma}) > 1$, then $\rho\left(\mathcal{T}_{\omega,\gamma}^{(1)}\right) > \rho(\mathcal{T}_{\omega,\gamma})$.*

Proof: From (7), we have

$$\begin{aligned}\mathcal{T}_{\omega,\gamma} = & \begin{pmatrix} \left(1-\frac{\gamma}{2}-\frac{\omega}{2}\right)I + \frac{\gamma+\omega}{2}B_1 & -\frac{\gamma+\omega}{2}D \\ \frac{\gamma+\omega}{2}\left(\frac{\omega}{2}-1\right)C_2 & \left(1-\frac{\gamma}{2}-\frac{\omega}{2}\right)I + \frac{\gamma+\omega}{2}B_2 \end{pmatrix} \\ & + \frac{(\gamma+\omega)\omega}{4}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_2B_1 & C_2D \end{pmatrix} \\ & + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \frac{\gamma+\omega}{2}\left(\frac{\gamma}{2}-1\right)C_1 & \mathbf{0} \end{pmatrix} + \frac{(\gamma+\omega)\gamma}{4}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_1B_1 & C_1D \end{pmatrix}.\end{aligned} \tag{11}$$

Since $D \le \mathbf{0}$, $B_1 \ge \mathbf{0}$, $B_2 \ge \mathbf{0}$, $C_1 \le \mathbf{0}$, $C_2 \le \mathbf{0}$, $0 < \omega + \gamma \le 2$, $\omega > 0$, $\gamma > 0$, it has

$$\begin{pmatrix} \left(1-\frac{\gamma}{2}-\frac{\omega}{2}\right)I + \frac{\gamma+\omega}{2}B_1 & -\frac{\gamma+\omega}{2}D \\ \frac{\gamma+\omega}{2}\left(\frac{\omega}{2}-1\right)C_2 & \left(1-\frac{\gamma}{2}-\frac{\omega}{2}\right)I + \frac{\gamma+\omega}{2}B_2 \end{pmatrix} \ge \mathbf{0},$$

$$\begin{aligned} & \frac{(\gamma+\omega)\omega}{4}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_2B_1 & C_2D \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \frac{\gamma+\omega}{2}\left(\frac{\gamma}{2}-1\right)C_1 & \mathbf{0} \end{pmatrix} \\ & + \frac{(\gamma+\omega)\gamma}{4}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -C_1B_1 & C_1D \end{pmatrix} \ge \mathbf{0},\end{aligned}$$

and the matrix $\mathcal{T}_{\omega,\gamma}$ is nonnegative. Since A is irreducible, from (7), it is easy to see that the matrix $\mathcal{T}_{\omega,\gamma}$ is also irreducible.

Similarly, since $b_{p1} > 0$, $\alpha > 0$, we have $S_1 \ge \mathbf{0}$, then $(I + S_1)D \le \mathbf{0}$, meanwhile, for $\alpha > 1 - b_{11}$, we have $B_1 - S_1(I - B_1) \ge \mathbf{0}$, so it can be proved that the matrix $\mathcal{T}_{\omega,\gamma}^{(1)}$ is nonnegative. The condition $\alpha > 1 - b_{11}$ ensures that the matrix $B_1 - S_1(I - B_1)$ has the same irreducibility as B_1, so the matrix $\mathcal{T}_{\omega,\gamma}^{(1)}$ is also nonnegative and irreducible.

By Lemma 2.1, there is a positive vector x, such that

$$\mathcal{T}_{\omega,\gamma}x = \lambda x, \tag{12}$$

where $\lambda = \rho(\mathcal{T}_{\omega,\gamma})$. Clearly, $\lambda = 1$ is impossible; otherwise, the matrix A is singular. Hence, it gets either $\lambda < 1$ or $\lambda > 1$.

Now, from (12) and by the definitions of $\mathcal{T}_{\omega,\gamma}$ and $\mathcal{T}_{\omega,\gamma}^{(1)}$, we have

$$\begin{aligned} & \mathcal{T}_{\omega,\gamma}^{(1)}x - \lambda x \\ = & \left(\mathcal{T}_{\omega,\gamma}^{(1)} - \mathcal{T}_{\omega,\gamma}\right)x \\ = & \begin{pmatrix} -\frac{\omega+\gamma}{2}S_1(I-B_1) & -\frac{\omega+\gamma}{2}S_1D \\ \mathcal{T}_{21}^{(1)} - \mathcal{T}_{21} & \mathcal{T}_{22}^{(1)} - \mathcal{T}_{22} \end{pmatrix}x \\ = & \begin{pmatrix} S_1 & \mathbf{0} \\ M_1 & \mathbf{0} \end{pmatrix}\begin{pmatrix} -\frac{\omega+\gamma}{2}(I-B_1) & -\frac{\omega+\gamma}{2}D \\ \mathbf{0} & \mathbf{0} \end{pmatrix}x \end{aligned}$$

$$= \begin{pmatrix} S_1 & \mathbf{0} \\ M_1 & \mathbf{0} \end{pmatrix} \begin{pmatrix} -\frac{\omega+\gamma}{2}(I-B_1) & -\frac{\omega+\gamma}{2}D \\ \mathcal{T}_{21} & \mathcal{T}_{22}-I \end{pmatrix} x$$

$$= \begin{pmatrix} S_1 & \mathbf{0} \\ M_1 & \mathbf{0} \end{pmatrix} \left[\begin{pmatrix} \left(1-\frac{\gamma}{2}-\frac{\omega}{2}\right)I+\frac{\omega+\gamma}{2}B_1 & -\frac{\omega+\gamma}{2}D \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{pmatrix} - \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} \right] x$$

$$= \begin{pmatrix} S_1 & \mathbf{0} \\ -\frac{1}{2}(\gamma C_1+\omega C_2)S_1 & \mathbf{0} \end{pmatrix} (\mathcal{T}_{\omega,\gamma}-I)\, x$$

$$= (\lambda-1) \begin{pmatrix} S_1 & \mathbf{0} \\ -\frac{1}{2}(\gamma C_1+\omega C_2)S_1 & \mathbf{0} \end{pmatrix} x$$

where

$$\mathcal{T}_{21}^{(1)} - \mathcal{T}_{21} = \frac{\gamma+\omega}{4}(\gamma C_1+\omega C_2)S_1(I-B_1),$$
$$\mathcal{T}_{22}^{(1)} - \mathcal{T}_{22} = \frac{\gamma+\omega}{4}(\gamma C_1+\omega C_2)S_1 D,$$
$$M_1 = -\frac{1}{2}(\gamma C_1+\omega C_2)S_1.$$

Since $C_1 \le \mathbf{0}$, $C_2 \le \mathbf{0}$, $\omega > 0$, $\gamma > 0$, $S_1 \ge \mathbf{0}$, $S_1 \ne \mathbf{0}$ and $x > \mathbf{0}$, it derives

$$\begin{pmatrix} S_1 & \mathbf{0} \\ -\frac{1}{2}(\gamma C_1+\omega C_2)S_1 & \mathbf{0} \end{pmatrix} x \ge \mathbf{0}, \quad \begin{pmatrix} S_1 & \mathbf{0} \\ -\frac{1}{2}(\gamma C_1+\omega C_2)S_1 & \mathbf{0} \end{pmatrix} x \ne \mathbf{0},$$

(i) If $\lambda < 1$, then $\mathcal{T}_{\omega,\gamma}^{(1)}x - \lambda x \le \mathbf{0}$, $\mathcal{T}_{\omega,\gamma}^{(1)}x - \lambda x \ne \mathbf{0}$. By Lemma 2.2, Theorem 2.1(i) is proved.
(ii) If $\lambda > 1$, then $\mathcal{T}_{\omega,\gamma}^{(1)}x - \lambda x \ge \mathbf{0}$, $\mathcal{T}_{\omega,\gamma}^{(1)}x - \lambda x \ne \mathbf{0}$. By Lemma 2.2, Theorem 2.1(ii) is proved.

Theorem 2.2. *Let $\mathcal{T}_{\omega,\gamma}$ and $\mathcal{T}_{\omega,\gamma}^{(2)}$ be the iteration matrices associated to the GTOR and PGTOR methods, respectively. If the matrix A in (1) is irreducible with $D \le \mathbf{0}$, $B_1 \ge \mathbf{0}$, $B_2 \ge \mathbf{0}$, $C_1 \le \mathbf{0}$, $C_2 \le \mathbf{0}$, $0 < \omega+\gamma \le 2$, $\omega > 0$, $\gamma > 0$, $\beta \in \left(-\frac{b_{p1}}{\alpha}, b_{p1}\left(\frac{1}{1-b_{11}} - \frac{1}{\alpha}\right)\right)$ when $1-b_{11} > 0$ or $\beta \in \left(-\frac{b_{p1}}{\alpha}, +\infty\right)$ when $1-b_{11} < 0$, then*

(i) *If $\rho(\mathcal{T}_{\omega,\gamma}) < 1$, then $\rho\left(\mathcal{T}_{\omega,\gamma}^{(2)}\right) < \rho(\mathcal{T}_{\omega,\gamma})$;*
(ii) *If $\rho(\mathcal{T}_{\omega,\gamma}) > 1$, then $\rho\left(\mathcal{T}_{\omega,\gamma}^{(2)}\right) > \rho(\mathcal{T}_{\omega,\gamma})$.*

Proof: When $D \le \mathbf{0}$, $B_1 \ge \mathbf{0}$, $B_2 \ge \mathbf{0}$, $C_1 \le \mathbf{0}$, $C_2 \le \mathbf{0}$, $0 < \omega+\gamma \le 2$, $\omega > 0$, $\gamma > 0$, $\mathcal{T}_{\omega,\gamma}$ is nonnegative and irreducible, it has been proved in Theorem 2.1.

Since $(I+S_2)D \le \mathbf{0}$, $B_1 - S_2(I-B_1) \ge \mathbf{0}$, $0 < \omega+\gamma \le 2$, it can be proved that the matrix $\mathcal{T}_{\omega,\gamma}^{(2)}$ is nonnegative. The condition $\beta \in \left(-\frac{b_{p1}}{\alpha}, b_{p1}\left(\frac{1}{1-b_{11}} - \frac{1}{\alpha}\right)\right)$ when $1-b_{11} > 0$ or $\beta \in \left(-\frac{b_{p1}}{\alpha}, +\infty\right)$ when $1-b_{11} < 0$ ensures that the matrix $B_1 - S_2(I-B_1)$ has the same irreducibility as B_1, so the matrix $\mathcal{T}_{\omega,\gamma}^{(2)}$ is also nonnegative and irreducible.

The rest proof is similar to the previous proof in Theorem 2.1.

Then we also give comparisons between different PGTOR methods.

Theorem 2.3. *Let $\mathcal{T}_{\omega,\gamma}$ be defined by (7), and $\mathcal{T}_{\omega,\gamma}^{(1)}$ and $\mathcal{T}_{\omega,\gamma}^{(2)}$ be defined by (10). Under the assumptions of Theorems 2.1 and 2.2, then*

(i) *If $\beta > 0$, $\rho(\mathcal{T}_{\omega,\gamma}) < 1$, then $\rho\left(\mathcal{T}_{\omega,\gamma}^{(1)}\right) > \rho\left(\mathcal{T}_{\omega,\gamma}^{(2)}\right)$;*
(ii) *If $\beta > 0$, $\rho(\mathcal{T}_{\omega,\gamma}) > 1$, then $\rho\left(\mathcal{T}_{\omega,\gamma}^{(1)}\right) < \rho\left(\mathcal{T}_{\omega,\gamma}^{(2)}\right)$;*

(iii) *If* $\beta < 0$, $\rho(\mathcal{T}_{\omega,\gamma}) < 1$, *then* $\rho\left(\mathcal{T}_{\omega,\gamma}^{(1)}\right) < \rho\left(\mathcal{T}_{\omega,\gamma}^{(2)}\right)$;

(iv) *If* $\beta < 0$, $\rho(\mathcal{T}_{\omega,\gamma}) > 1$, *then* $\rho\left(\mathcal{T}_{\omega,\gamma}^{(1)}\right) > \rho\left(\mathcal{T}_{\omega,\gamma}^{(2)}\right)$.

Proof: The given conditions imply that the matrices $\mathcal{T}_{\omega,\gamma}$, $\mathcal{T}_{\omega,\gamma}^{(1)}$ and $\mathcal{T}_{\omega,\gamma}^{(2)}$ are nonnegative and irreducible. By Lemma 2.1 and the definitions of $\mathcal{T}_{\omega,\gamma}^{(1)}$ and $\mathcal{T}_{\omega,\gamma}^{(2)}$, we have

$$\begin{aligned}
&\mathcal{T}_{\omega,\gamma}^{(1)}x - \mathcal{T}_{\omega,\gamma}^{(2)}x \\
&= \left(\mathcal{T}_{\omega,\gamma}^{(1)}x - \lambda x\right) - \left(\mathcal{T}_{\omega,\gamma}^{(2)}x - \lambda x\right) \\
&= \left[\mathcal{L}_{\omega,\gamma}^{(1)} - \mathcal{L}_{\omega,\gamma}\right]x - \left[\mathcal{L}_{\omega,\gamma}^{(2)} - \mathcal{L}_{\omega,\gamma}\right]x \\
&= (\lambda - 1)\begin{pmatrix} S_1 & \mathbf{0} \\ -\frac{1}{2}(\gamma C_1 + \omega C_2)S_1 & \mathbf{0} \end{pmatrix} x - (\lambda - 1)\begin{pmatrix} S_2 & \mathbf{0} \\ -\frac{1}{2}(\gamma C_1 + \omega C_2)S_2 & \mathbf{0} \end{pmatrix} x \\
&= (\lambda - 1)\begin{pmatrix} I & \mathbf{0} \\ -\frac{1}{2}(\gamma C_1 + \omega C_2) & \mathbf{0} \end{pmatrix}(S_1 - S_2)x
\end{aligned}$$

Under the conditions of Theorem 2.1 and Lemma 2.1, we can know that

$$\begin{pmatrix} I & \mathbf{0} \\ -\frac{1}{2}(\gamma C_1 + \omega C_2) & \mathbf{0} \end{pmatrix} \geq \mathbf{0}, \quad x > \mathbf{0},$$

meanwhile, it is easy to know that

When $\beta > 0$, $\mathbf{0} \leq S_1 \leq S_2$ and $S_1 \neq S_2$, thus $S_1 - S_2 \leq 0$, $S_1 - S_2 \neq 0$,

(i) If $\lambda < 1$, then $\mathcal{T}_{\omega,\gamma}^{(1)}x - \mathcal{T}_{\omega,\gamma}^{(2)}x \geq \mathbf{0}$, $\mathcal{T}_{\omega,\gamma}^{(1)}x - \mathcal{T}_{\omega,\gamma}^{(2)}x \neq \mathbf{0}$.
By Lemma 2.2, Theorem 2.3(i) is proved.

(ii) If $\lambda > 1$, then $\mathcal{T}_{\omega,\gamma}^{(1)}x - \mathcal{T}_{\omega,\gamma}^{(2)}x \leq \mathbf{0}$, $\mathcal{T}_{\omega,\gamma}^{(1)}x - \mathcal{T}_{\omega,\gamma}^{(2)}x \neq \mathbf{0}$.
By Lemma 2.2, Theorem 2.3(ii) is proved.

When $\beta < 0$, $S_1 \geq S_2 \geq 0$ and $S_1 \neq S_2$, thus $S_1 - S_2 \geq 0$, $S_1 - S_2 \neq 0$,

(iii) If $\lambda < 1$, then $\mathcal{T}_{\omega,\gamma}^{(1)}x - \mathcal{T}_{\omega,\gamma}^{(2)}x \leq \mathbf{0}$, $\mathcal{T}_{\omega,\gamma}^{(1)}x - \mathcal{T}_{\omega,\gamma}^{(2)}x \neq \mathbf{0}$.
By Lemma 2.2, Theorem 2.3(iii) is proved.

(iv) If $\lambda > 1$, then $\mathcal{T}_{\omega,\gamma}^{(1)}x - \mathcal{T}_{\omega,\gamma}^{(2)}x \geq \mathbf{0}$, $\mathcal{T}_{\omega,\gamma}^{(1)}x - \mathcal{T}_{\omega,\gamma}^{(2)}x \neq \mathbf{0}$.
By Lemma 2.2, Theorem 2.3(iv) is proved.

3. **Algorithms.**

Based on the above analysis, the resulting algorithms are summarized as follows.

Algorithm 1 (GTOR algorithm for solving (1)).
(1) Input n, ω, γ, p. Set $x^{(0)} = \mathbf{0}$;
(2) Compute $x^{(k+1)}$ by (3);
(3) Stop if the stopping criteria $\left\|x^{(k+1)} - x^{(k)}\right\| < tol$ or $k >$ max are satisfied; otherwise, set $k := k + 1$, go to step (2).

Algorithm 2 (PGTORi ($i = 1, 2$) algorithms for solving (2)).
(1) Input n, ω, γ, p, α, β. Set $x^{(0)} = \mathbf{0}$;
(2) Compute $x^{(k+1)}$ by (8). Set $i = 1, 2$, respectively;
(3) Stop if the stopping criteria $\left\|x^{(k+1)} - x^{(k)}\right\| < tol$ or $k >$ max are satisfied; otherwise, set $k := k + 1$, go to step (2).

4. **Numerical Example.**

In all cases, all iterations were started from the zero initial vector and terminated when $\left\|x^{(k+1)} - x^{(k)}\right\|_\infty < 10^{-9}$, where $x^{(k)}$ denotes the kth iterative vector for the corresponding iterative method. The maximum number of iterations of all tests was 1000. All calculation results were obtained using the TOSHIBA computer of which the CPU is Inter(R) Core(TM) 2 Duo, the computer memory is 2 G and the operating system is Win7, and all test programs were written in Matlab 7.9.

Example 4.1. *Consider the following boundary value problem*

$$\Delta u = 0, \quad 0 < x < 2, \quad 0 < y < 2 \tag{13}$$

with boundary conditions

$$u(0,y) = 0, \quad u(2,y) = y(2-y), \quad 0 < y < 2$$

$$u(x,0) = 0, \quad u(x,2) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \end{cases}$$

When the central difference scheme on a uniform grid with $N \times N$ interior nodes ($N^2 = n$) is applied to this Equation (13), we can obtain a system of linear equations (1). The test matrix A and vector b arise from five-point discretization of the second order PDE $\Delta u = 0$.

Then we test for the GTOR iterative method and its two preconditioned inversions. In Table 1, GTOR stands for the GTOR method, PGTORi represent the PGTORi methods which are with preconditioned matrix $I + S_i$ ($i = 1, 2$), respectively. We denote the number of iterations by IT, and the program execution time by CPU. The parameters n, ω, γ, p, α and β are randomly chosen parameters which meet the conditions of former theorems and corollaries in Section 3 when we choose $C_1 = \frac{2}{3}C$, $C_2 = \frac{1}{3}C$.

Table 1. CPU time and the number of iterations of the GTOR algorithm and two preconditioned GTOR algorithms

n	ω	γ	p	α	β	GTOR		PGTOR1		PGTOR2	
						IT	CPU	IT	CPU	IT	CPU
400	0.8	0.5	100	2	1	534	0.359	534	0.328	516	0.317
900	0.8	0.5	300	3	2	523	3.338	523	3.119	507	2.808
1600	0.8	0.5	500	4	2	510	14.601	510	13.184	498	12.251
2500	0.8	0.5	1000	3	1	502	48.407	502	41.532	497	38.381

From Table 1, we can see that the PGTORi ($i = 1, 2$) algorithms are better than the GTOR algorithm. In addition, the PGTOR2 algorithm is better than the PGTOR1 algorithm. Especially when the order of matrix A (namely n) is large enough, the time advantage of the former methods is obvious. These conclusions are in accordance with the theoretical results in Section 2.

5. Conclusions.

In this paper, we have provided comparison results of several types of PGTOR methods for solving a linear system (1). What really sets the matrix A and the preconditioners apart is the block structure (6) which is seldom considered before, and we can also consider the other three $\widetilde{S}$, with S lying in other locations $\widetilde{S} = \begin{pmatrix} \mathbf{0} & S \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, $\widetilde{S} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ S & \mathbf{0} \end{pmatrix}$, $\widetilde{S} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & S \end{pmatrix}$. Furthermore, it is difficult to select the optimal parameters of ω, γ, α and β. These all require further study.

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