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STATE ESTIMATION FOR NETWORKED CONTROL SYSTEMS AT THE EXTREME DATA-RATE LIMIT

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Abstract. *This paper investigates state estimation for linear discrete-time control systems where sensors and controllers are geographically separated and connected via a bandwidth-limited communication channel. In particular, we consider the case at the extreme data-rate limit (namely, when the data rate decreases to its theoretical minimum value). In this case, it is possible to ensure observability of the system without the disturbances when system matrix has only real eigenvalues each with geometric multiplicity one. Conversely, observability is lost if the system cannot be decoupled, and the plant states are mutually related. The illustrative examples are given to demonstrate the effectiveness of the proposed scheme.*
Keywords: State estimation, Data-rate limit, Observability, Networked control systems

1. **Introduction.** Communication constraints arise when control systems employ non-transparent communication links [1]. Such constraints include mainly random delays, packet drop and data-rate limits [2]. In this paper, we are interested in data-rate constraints.

Control under communication constraints inevitably suffers signal transmission delay, data packet dropout and measurement quantization which might be potential sources of instability and poor performance of control systems [3]. [4] investigated the quantized feedback control problem for stochastic time-invariant linear control systems. A predictive control policy under data-rate constraints was proposed to stabilize the unstable plant in the mean square sense. [5] addressed LQ (linear quadratic) control of MIMO (multi-input multi-output), discrete-time linear systems, and gave the inherent tradeoffs between LQ cost and data rates. In [6], a quantized-observer based encoding-decoding scheme was designed, which integrated the state observation with encoding-decoding. [7] addressed some of the challenging issues on moving horizon state estimation for networked control systems in the presence of multiple packet dropouts. The recent paper by Nair [8] addressed problems of information theory for nonstochastic variables and state estimation for networked control systems. In particular, [8] proposed a formal framework for modelling nonstochastic uncertain variables, and derived a necessary and sufficient condition on zero-error capacity [9], and [10] for state estimation.

A high-water mark in the study of quantized feedback using data-rate limited feedback channels is known as the data-rate theorem [1]. This states that linear control systems are stabilizable or observable if the data rate of communication channels is larger than a lower bound given. This result establishes some essential requirements for stability of networked control systems [1]. However, it is still unclear whether such systems are stabilizable or observable when the data rate decreases to its theoretical minimum value.

In this paper, we focus on conditions on the data rate for observability of linear discrete-time control systems, deal with the case at the extreme data-rate limit, and further develop the data-rate theorem.

The remainder of this paper is organized as follows: Section 2 introduces problem formulation; Section 3 deals with state estimation at the extreme data-rate limit; the results of numerical simulation are presented in Section 4; conclusions are stated in Section 5.

2. Problem Formulation.

Let us consider a linear time-invariant system

$$\begin{aligned} X(k+1) &= AX(k) + BU(k) \in \mathbb{R}^n, \\ Y(k) &= CX(k) \in \mathbb{R}^p,\ k \in \mathbb{Z}_{\geq 0} \end{aligned} \tag{1}$$

where $X(k) \in \mathbb{R}^n$ denotes the plant state, $U(k) \in \mathbb{R}^q$ denotes the control input, and $Y(k) \in \mathbb{R}^p$ denotes the measured output. A, B, and C are known constant matrices with appropriate dimensions. Without loss of generality, we assume that the pair (A, C) is observable and the pair (A, B) is controllable. Furthermore, we assume that the initial state $X(0)$ is a bounded, uncertain variable satisfying $\|X(0)\| \leq \phi_0 < \infty$.

The output signal $Y(k)$ is causally encoded via an operator Θ as

$$C(k) = \Theta(k, Y(0:k)),\ k \in \mathbb{Z}_{\geq 0}. \tag{2}$$

Each symbol $C(k)$ is transmitted over a stationary memoryless uncertain channel without channel feedback. $D(k)$ denotes the received symbol. These received symbols are used to produce a causal prediction $\hat{X}(k+1)$ of $X(k+1)$ by means of another operator Ψ as

$$\hat{X}(k+1) = \Psi(k, D(0:k)),\ k \in \mathbb{Z}_{\geq 0},\ \hat{X}(0) = 0. \tag{3}$$

The pair (Θ, Ψ) is called a coder-estimator.

Let $E(k) := X(k) - \hat{X}(k)$ denote the prediction error. The system (1) is said to be observable if the estimation error is bounded for any time k. Namely,

$$\limsup_{k\to\infty} \|E(k)\| < \infty$$

holds for any initial state $X(0)$ satisfying $\|X(0)\| \leq \phi_0 < \infty$.

As stated in [1], a high-water mark in the study of quantized feedback using data-rate limited feedback channels is known as the data-rate theorem. Let λ_i denote the ith eigenvalue of system matrix A ($i = 1, 2, \ldots, n$). It was shown in the data-rate theorem that, the system (1) is stabilizable or observable if the data rate R of communication channels satisfies the following inequality:

$$R > \sum_{i\in\Xi} \log_2 |\lambda_i|$$

with $\Xi := \{i \in \{1, 2, \ldots, n\} : |\lambda_i| \geq 1\}$. In this paper, the difference is that we argue about the state estimation problem for linear time-invariant systems when the data rate R is equal to the lower bound. Namely, we deal with the case at the extreme data-rate limit, and further develop the data-rate theorem. Then, the main problem here is to derive the condition for observability of the system (1) when the data rate decreases to its theoretical minimum value.

3. Observability at the Extreme Data-Rate Limit.

This section deals with the state estimation problem at the extreme data-rate limit. Here, it is assumed that there exists a real-valued nonsingular matrix H and a real-valued matrix Λ such that $\Lambda = HAH' = diag[J_1, \ldots, J_m]$ holds, where each J_j, $j = 1, \ldots, m$, is a Jordan block of dimension (geometric multiplicity) n_j. Here, we define $\bar{X}(k) := HX(k)$, $\tilde{X}(k) := H\hat{X}(k)$, and $\bar{E}(k) := HE(k)$. Then, the system (1) can be written as

$$\begin{aligned} \bar{X}(k+1) &= \Lambda\bar{X}(k) + HBU(k) \in \mathbb{R}^n, \\ Y(k) &= CH'\bar{X}(k) \in \mathbb{R}^p,\ k \in \mathbb{Z}_{\geq 0}. \end{aligned} \tag{4}$$

First, we consider the case with $n_j = 1$ $(j = 1, \ldots, m)$. Namely, all the eigenvalues of system matrix A are distinct. Then, we give the following result.

Theorem 3.1. *Consider the system (1) with the coder (2) and estimator (3). It is assumed that there exists a real-valued nonsingular matrix H and a real-valued matrix Λ such that $\Lambda = HAH' = diag[\lambda_1, \ldots, \lambda_n]$ holds. Then, the system (1) is observable if the data rate R of communication channels satisfies the following equality:*

$$R = \sum_{i\in\Xi} \log_2 |\lambda_i|$$

with $\Xi := \{i \in \{1, 2, \ldots, n\} : |\lambda_i| \geq 1\}$.

Proof: Let $B_l(c)$ denote the range $\{x : |x - c| < l\}$ centered at c. We define

$$\bar{X}(k) := [\bar{x}_1(k)\ \bar{x}_2(k) \cdots \bar{x}_n(k)]',$$
$$\tilde{X}(k) := [\tilde{x}_1(k)\ \tilde{x}_2(k) \cdots \tilde{x}_n(k)]',$$
$$\bar{E}(k) := [\bar{e}_1(k)\ \bar{e}_2(k) \cdots \bar{e}_n(k)]'.$$

By the assumption, we know that

$$\bar{x}_i(0) \in B_{\|H\|\phi_0}(0), \quad i = 1, 2, \ldots, n.$$

Let r_i denote the data rate corresponding to $\bar{x}_i(k)$. If we assume that

$$\bar{x}_i(k) \in B_{l_i(k)}(c_i(k)),$$
$$\tilde{x}_i(k) = c_i(k),$$
$$\bar{e}_i(k) \in B_{l_i(k)}(0), \quad i = 1, 2, \ldots, n,$$

it follows that

$$\bar{x}_i(k+1) \in B_{l_i(k+1)}(c_i(k+1)),$$
$$\tilde{x}_i(k+1) = c_i(k+1),$$
$$\bar{e}_i(k+1) \in B_{l_i(k+1)}(0), \quad i = 1, 2, \ldots, n,$$

where

$$l_i(k+1) = \begin{cases} \frac{|\lambda_i|}{2^{r_i}} l_i(k), & \text{when } |\lambda_i| \geq 1 \\ |\lambda_i| l_i(k), & \text{when } |\lambda_i| < 1 \end{cases}$$

Then, we have

$$l_i(k) = \begin{cases} \left(\frac{|\lambda_i|}{2^{r_i}}\right)^k \|H\|\phi_0, & \text{when } |\lambda_i| \geq 1 \\ (|\lambda_i|)^k \|H\|\phi_0, & \text{when } |\lambda_i| < 1 \end{cases} \tag{5}$$

Clearly, it follows from (5) that

$$\lim_{k\to\infty} l_i(k) = \begin{cases} \|H\|\phi_0, & \text{when } |\lambda_i| \geq 1 \\ 0, & \text{when } |\lambda_i| < 1 \end{cases} \tag{6}$$

if the data rate r_i satisfies the following equality:

$$r_i = \log_2 |\lambda_i| \text{ (bits/sample)}.$$

Therefore, this gives

$$\limsup_{k\to\infty} \|E(k)\| < \phi_0 < \infty,$$

if the data rate R satisfies the following equality:

$$R = \sum_{i\in\Xi} \log_2 |\lambda_i| \text{ (bits/sample)}.$$

The proof is complete. □

It is shown in Theorem 3.1 that the system (1) is observable at the extreme data-rate limit if all the eigenvalues of system matrix A are distinct. However, that is a special case. Next, we address state estimation for general case, and give the following result.

Theorem 3.2. *Consider the system (1) with the coder (2) and estimator (3). It is assumed that there exists a real-valued nonsingular matrix H and a real-valued matrix Λ such that $\Lambda = HAH' = diag[J_1, \ldots, J_m]$ holds, where each J_j, $j = 1, \ldots, m$, is a Jordan block of dimension (geometric multiplicity) n_j. Suppose that at least one n_j is larger than 1 and the corresponding eigenvalue has magnitude larger than 1. For the system (1), the estimation error of the plant state is unbounded if the data rate R of communication channels satisfies the following equality:*

$$R = \sum_{i\in\Xi} \log_2 |\lambda_i|$$

with $\Xi := \{i \in \{1, 2, \ldots, n\} : |\lambda_i| \geq 1\}$.

Proof: Using the same techniques as the proof of Theorem 3.1, we can show that for the case with $n_j = 1$,

$$l_i(k) = \begin{cases} \left(\frac{|\lambda_i|}{2^{r_i}}\right)^k \|H\|\phi_0, & \text{when } |\lambda_i| \geq 1 \\ (|\lambda_i|)^k \|H\|\phi_0, & \text{when } |\lambda_i| < 1 \end{cases} \tag{7}$$

and for the case with $n_j > 1$,

$$l_i(k) = \begin{cases} \left(\frac{|\lambda_i|}{2^{r_i}}\right)^k \|H\|\phi_0 + \sum_{d=0}^{k-1} \left(\frac{|\lambda_i|}{2^{r_i}}\right)^d l_{i+1}(k-1-d), & \text{when } |\lambda_i| \geq 1 \\ (|\lambda_i|)^k \|H\|\phi_0 + \sum_{d=0}^{k-1} (|\lambda_i|)^d l_{i+1}(k-1-d), & \text{when } |\lambda_i| < 1 \end{cases} \tag{8}$$

Thus, it follows from (7) and (8) that for the case with $n_j = 1$,

$$\lim_{k\to\infty} l_i(k) = \begin{cases} \|H\|\phi_0, & \text{when } |\lambda_i| \geq 1 \\ 0, & \text{when } |\lambda_i| < 1 \end{cases}$$

and for the case with $n_j > 1$

$$\lim_{k\to\infty} l_i(k) = \begin{cases} \infty, & \text{when } |\lambda_i| \geq 1 \\ 0, & \text{when } |\lambda_i| < 1 \end{cases}$$

if the data rate r_i satisfies the following equality:

$$r_i = \log_2 |\lambda_i| \text{ (bits/sample)}.$$

Therefore, this gives

$$\limsup_{k\to\infty} \|E(k)\| = \infty,$$

if the data rate R satisfies the following equality:

$$R = \sum_{i\in\Xi} \log_2 |\lambda_i| \text{ (bits/sample)}.$$

The proof is complete. □

4. **Numerical Examples.**

This section contains numerical examples that illustrate the utility of the results of Section 3.

Example 4.1. *Consider the discrete-time control system*

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

where $\lambda_1 = 4$ *and* $\lambda_2 = 8$. *Here, we assume that* $x_1(0) \in B_2(0)$ *and* $x_2(0) \in B_4(0)$. *Without loss of generality, we set* $x_1(0) = 1.4$ *and* $x_2(0) = 2.6$, *respectively.*

In this example, we examine the state estimation problem at the extreme data-rate limit. Here, we set $R = 5$ (bits/sample). The corresponding simulation is given in Figure 1. Here, $\bar{e}_1(k)$ and $\bar{e}_2(k)$ denote the estimation errors of $x_1(k)$ and $x_2(k)$, respectively. It can be seen that the estimation errors are bounded as $k \to \infty$.

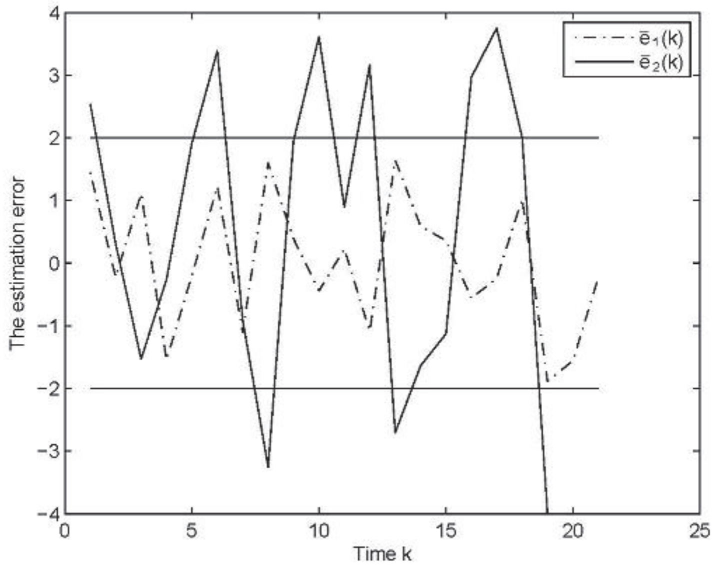

Figure 1. The initial condition response of the system in Example 4.1

Example 4.2. *Consider the discrete-time control system*

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

where $\lambda_1 = \lambda_2 = 4$. Here, we assume that $x_1(0) \in B_2(0)$ and $x_2(0) \in B_4(0)$. In this example, we examine the state estimation problem at the extreme data-rate limit when system matrix has eigenvalue with geometric multiplicity larger than one. Here, we set $R = 4$ (bits/sample). The corresponding simulation is given in Figure 2. Here, $\bar{e}_1(k)$ and $\bar{e}_2(k)$ denote the estimation errors of $x_1(k)$ and $x_2(k)$, respectively. It can be seen that the estimation errors are unbounded as $k \to \infty$.

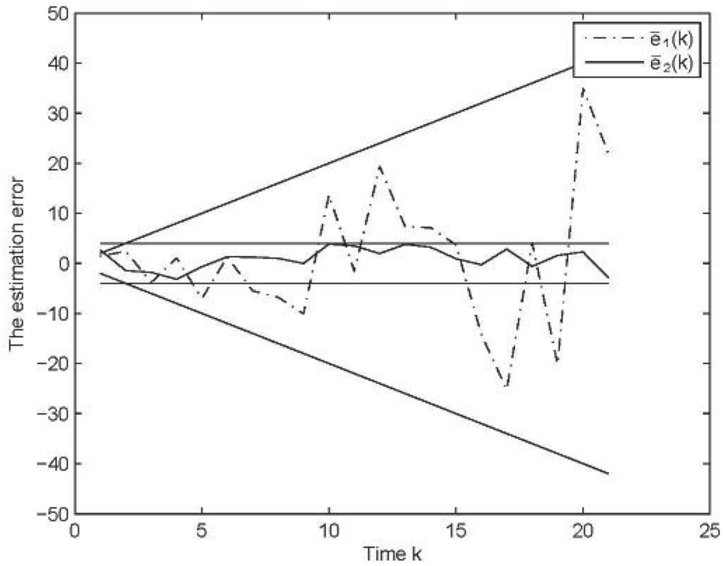

Figure 2. The initial condition response of the system in Example 4.2

5. **Conclusions.** In this paper, we considered linear discrete-time control systems with data-rate constraints, argued about the state estimation problem when the data rate decreased to its theoretical minimum value, and presented conditions for observability at the extreme data-rate limit. The simulation results have illustrated the effectiveness of the proposed scheme. The study of nonlinear system with limited information rates will be our future work.

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