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AN INVENTORY PROBLEM WITH SPACE LIMIT AND INCREASING SHIPPING BATCH SIZE

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Abstract. *Production-inventory cooperation model with one buyer and one vendor is studied. If the buyer places an order with lots of products, then the vendor begins to produce the products and these will be transferred to the buyer with unequal-sized shipments many times. Actually, shipping batch size is increased with fixed factor and space restrictions for vendor and buyer are considered. Demand follows normal distribution and the continuous review inventory policy is used to the buyer. A Lagrangian relaxation approach is developed to find lower bound of this problem and the solution method based on the Lagrangian relaxation approach is proposed. Sensitivity analysis is done to figure out the relationships between space restriction and decision variables such as order quantity, safety factor, the number of shipments, and fixed factor for shipping batch size increase.*
Keywords: Continuous review inventory system, Increasing shipping batch, Space restriction, Supply chain management

1. **Introduction.** The cooperation among companies of the various cost reduction activities is more and more important in supply chain. In this research, an integrated inventory strategy through the cooperation between a vendor and a buyer is studied. To consider realistic situation, we include space limit for vendor and buyer.

There are some researches on continuous review inventory model. Tamjidzad and Mirmohammadi [11] develop the optimal solution algorithm for continuous review inventory problem with quantity discount and limited sharable resource. The problem is broken down into sub-problems and each sub-problem is solved optimally. Alfares and Ghaithan [1] define the inventory model with the variable demand rate, holding cost, and purchase cost. They prove that their objective function is concave and develop the optimal solution algorithm. Ghalebsaz-Jeddi et al. [3], Wang and Hu [8,9], and Zhao et al. [10] propose the solution method for multi-item continuous review inventory with space limit. Hariga [6] studies that a buyer uses continuous review inventory policy and his storage space is limited. If the inventory is over storage capacity, then the buyer returns the inventory over space capacity to the buyer and pays extra charges to the vendor.

On the other hand, production-inventory cooperation model with one buyer and one vendor is studied in some researches. Goyal [5] deals with this problem and presents a solution method for minimizing inventory cost if customer's demand is deterministic and the sizes of shipping batches are equal. Hill [7] provides the optimal solution approach if customer's demand is deterministic and the sizes of shipping batches are increasing with a fixed factor. Ben-Daya and Hariga [2] suggest a solution approach if demand is stochastic and the sizes of shipping batches are equal. Glock [4] presents the optimal solution approach if customer's demand is probabilistic and the sizes of shipping batches are increasing with a fixed factor.

Space restriction is not considered in these researches. In real situation, space restriction is required for inventory model because the size of warehouse is limited. In this paper, production-inventory cooperation model with stochastic demand and increasing shipping batch size is studied. Space restriction is considered and the space limit constraint is defined in mathematical formulation. A Lagrangian relaxation approach is developed to minimize total cost.

This paper is organized as follows. Section 2 provides problem definition and mathematical formulation. In Section 3, the first order necessary condition and the solution method based Lagrangian relaxation method is proposed. In Section 4, numerical example is illustrated and sensitivity analysis is conducted to reveal the relationship between the space limit and decision variables. Finally, in Section 5, the conclusion is presented.

2. **Problem Definition.** Buyer's demand is stochastic and follows normal distribution. Vendor uses a continuous inventory review (Q, r) policy. If shipping bathes increase with a fixed factor α (≥ 1), then the size of shipping batch in the jth shipment will be $q_j = q_1\alpha^{j-1}$ where q_1 is the first shipping batch size. Lead time is proportional to vendor's lot size plus a fixed delayed time (b) such as transportation and non-production time. Therefore, lead time will be $LT(q_j) = pq_j + b$. Buyer places the number of $\sum_{j=1}^{n} q_j$ of the item, and vendor manufacture $\sum_{j=1}^{n} q_j$ of that item with annual production rate $1/p$. Since buyer uses (Q, r) inventory policy, the buyer receives n number of shipments from vendor and the size of shipping batch is q_j in the jth shipment. The buyer orders again if his on hand inventory level hits a reorder point r, right after getting the nth shipment.

Notations and variables in this problem are defined as follows.

W_v: maximum allowable vendor's inventory level

W_b: maximum allowable buyer's inventory level

Z: random variable for standard normal distribution, $N(0, 1)$

X: random variable during lead time demand, $X \sim N(\mu, \sigma^2)$

$L(r)$: expected shortage demand for reorder point r, $L(r) = \int_r^{\infty}(x - r)f(x)dx$

$f(x)$: probability density function (p.d.f) for random variable X

π: shortage cost per item for buyer

h_v: inventory holding cost per item for vendor

h_b: inventory holding cost per item for buyer

A_v: setup cost for vendor per order

A_b: setup cost for buyer per order

F: transportation cost for buyer per shipment

D: annual demand

Decision Variables. q_1: size of the first shipment from the vendor to the buyer; z_1: safety factor for the buyer in the first shipment; n: number of shipments; α: fixed factor for shipping batch increase.

We can formulate this problem (P) in the following.

$$\begin{aligned}
\mathrm{P}: \quad & \mathrm{Min}\ TC(q_1, z_1, n, \alpha) \\
= & \frac{q_1}{2}\left[\left(2Dp + (1 - Dp)\sum_{i=1}^{n}\alpha^{i-1}\right)h_v + \left(\frac{\sum\limits_{i=1}^{n}\alpha^{2i-2}}{\sum\limits_{j=1}^{n}\alpha^{j-1}}\right)(h_b - h_v)\right] \\
& + z_1\sigma\sqrt{pq_1 + b}h_b + \left[(A_b + A_v + nF)\right. \\
& \left. + n\sigma\sum_{i=1}^{n}\sqrt{pq_1\alpha^{i-1} + b}L(z_i)\right]\frac{D}{q_1\sum\limits_{i=1}^{n}\alpha^{i-1}} \qquad (1)
\end{aligned}$$

subject to

$$q_1\alpha^{n-1} + z_1\sigma\sqrt{pq_1 + b} \leq W_b \tag{2}$$

$$q_1\alpha^{n-1} \leq W_v \tag{3}$$

$$q_1, z_1 \geq 0 \tag{4}$$

$$n \in N \tag{5}$$

(1) represents the objective function that minimizes total cost of vendor and buyer. (2) ensures that buyer's maximum inventory level cannot exceed maximum allowable inventory level (W_b). Since the last shipment $q_1\alpha^{n-1}$ is the largest shipment for the buyer, buyer's maximum inventory level can be represented by the sum of the last shipment and safety stock, $q_1\alpha^{n-1} + z_1\sigma\sqrt{pq_1 + b}$. By the same token, (3) ensures that vendor's least maximum inventory level, $q_1\alpha^{n-1}$, cannot exceed W_v. Vendor's maximum inventory level is not easily obtained, and vendor's maximum inventory level is at least larger than the size of the last shipment, $q_1\alpha^{n-1}$. We use vendor's least maximum inventory level, $q_1\alpha^{n-1}$. (4) defines decision variables q_1, z_1 are positive real numbers and (5) defines the decision variable n is a positive integer.

3. **The Proposed Method.** Lagrangian relaxation problem for primal problem (P) is defined, and the first order necessary condition is introduced.

3.1. **Lagrangian relaxation and the first order necessary condition.** We relax space restriction constraints for vendor and buyer (2), (3) and use associated Lagrange multipliers λ_v, λ_b to formulate Lagrangian dual problem:

$$\begin{aligned} \text{LD}: \quad & \text{Max}_{\lambda_v,\lambda_b}\ \text{Min}L \\ = \ & TC(q_1, z_1, n, \alpha) - \lambda_v\left(W_v - q_1\alpha^{n-1}\right) - \lambda_b\left(W_b - q_1\alpha^{n-1} - z_1\sigma\sqrt{pq_1 + b}\right) \\ & \text{Subject to (4)} \end{aligned}$$

n is relaxed to be nonnegative real number. The optimal solution of LD, $\text{Max}_{\lambda_v,\lambda_b}$ is the greatest lower bound of P. Given that λ_v, λ_b are determined, the Lagrangian relaxed problem (LR) is shown by

$$\begin{aligned} \text{LR}: \text{Min } L \ = \ & \frac{q_1}{2}\left[\left(2Dp + (1 - Dp)\sum_{i=1}^{n}\alpha^{i-1}\right)h_v + \left(\frac{\sum\limits_{i=1}^{n}\alpha^{2i-2}}{\sum\limits_{i=1}^{n}\alpha^{i-1}}\right)(h_b - h_v)\right] \\ & + z_1\sigma\sqrt{pq_1 + b}h_b \\ & + \left[(A_b + A_v + nF) + \pi\sigma\sum_{i=1}^{n}\sqrt{pq_1\alpha^{i-1} + b}L(z_i)\right]\frac{D}{q_1\sum\limits_{i=1}^{n}\alpha^{i-1}} \\ & + \lambda_v\left(q_1\alpha^{n-1} - W_v\right) + \lambda_b\left(q_1\alpha^{n-1} + z_1\sigma\sqrt{pq_1 + b} - W_b\right) \\ & \text{Subject to (4)} \end{aligned} \tag{6}$$

Differentiating (6) with respect to q_1, z_1, λ_v, λ_b, and set those to 0, then we can obtain the first order necessary conditions for LR as follows:

$$q_1 = \sqrt{\frac{Q_N}{Q_D}} \tag{7}$$

where, $Q_N = 2D\left[(A_b + A_v + nF) + \pi\sigma \sum_{i=1}^n \sqrt{pq_1\alpha^{i-1} + b}L(z_i)\right]$, $Q_D = \sum_{i=1}^n \alpha^{i-1}\left[C + \frac{p\sigma z_1}{\sqrt{pq_1+b}}(h_b + \lambda_b) + \pi\sigma\sum_{i=1}^n\left(\frac{Dp\alpha^{i-1}L(z_i)}{q_1\sum_{i=1}^n \alpha^{i-1}\sqrt{pq_1\alpha^{i-1}+b}}\right) + 2\alpha^{n-1}(\lambda_v + \lambda_b)\right]$, and $C = \left[\left(2Dp + (1-Dp)\sum_{i=1}^n \alpha^{i-1}\right)h_v + \left(\frac{\sum_{i=1}^n \alpha^{2i-2}}{\sum_{i=1}^n \alpha^{i-1}}\right)(h_b - h_v)\right]$.

$$\frac{q_1(h_b + \lambda_b)}{\pi D} = \frac{1}{\sum_{i=1}^n \alpha^{i-1}}\sum_{i=1}^n \sqrt{\frac{pq_1\alpha^{i-1} + b}{pq_1 + b}}(1 - F(z_i)) \tag{8}$$

$$\frac{\partial L}{\partial \lambda_v} = q_1\alpha^{n-1} - W_v = 0 \tag{9}$$

$$\frac{\partial L}{\partial \lambda_b} = q_1\alpha^{n-1} + z_1\sigma\sqrt{pq_1 + b} - W_b = 0 \tag{10}$$

Note that the first necessary condition for decision variable n is not considered. n is a positive integer and n will be found by total enumeration in the next section.

3.2. **Solution methodology.** Since Equations (7) and (8) are influenced by λ_v, λ_b, constraints (2) and (3) can be represented by functions of λ_v, λ_b as follows:

$$g_v(\lambda_v) = q_1\alpha^{n-1} - W_v \tag{11}$$

$$g_b(\lambda_b) = q_1\alpha^{n-1} + z_1\sigma\sqrt{pq_1 + b} - W_b \tag{12}$$

If $g_v, g_b \le 0$, then constraints (2) and (3) are not violated.

For given λ_v, λ_b, the procedure for LR is to find q_1, z_i that satisfy both of (7) and (8) while increasing n by 1. Then, search α to minimize L. Repeat this until the objective value L is not decreased. The proposed method for LR is listed in the following.

Algorithm A. The algorithm for getting q_1, z_i, n, α, the solution of Problem (LR)

Step 1. For given λ_v, λ_b, set $L(z_i) = 0$, $n = 1$.

Step 2. Set $\alpha = 1$.

Step 3. Repeat a) and b) until q_1 and z_i converge.

a) Get q_1 by inserting z_i into (7).

b) Get z_i by inserting q_1 into (8).

Step 4. Update α by golden section search. If α does not change, then go to Step 5. Otherwise, return to Step 3. Note that maximum value of α is pD.

Step 5. If TC decreases, then set $n = n + 1$ and return to Step 2. Otherwise, stop.

Solutions for Problem LR may be infeasible solutions for Problem P. To make those feasible, we need to find q_1 such that satisfy constraints (2) and (3) at the same time. First, find Q_v satisfying $g_v(\lambda_v) = 0$, and Q_b satisfying $g_b(\lambda_b) = 0$. Then, we can choose minimum between Q_v and Q_b so that both constraints are satisfied.

If $g_v(\lambda_v) = 0$, then $q_1\alpha^{n-1} = W_v$. Therefore, $Q_v = \frac{W_v}{\alpha^{n-1}}$. If $g_b(\lambda_b) = 0$, then $q_1\alpha^{n-1} + z_1\sigma\sqrt{pq_1 + b} = W_b$. Therefore, $Q_b = \frac{W_b - z_q\sigma\sqrt{pq_1+b}}{\alpha^{n-1}}$. Solve iteratively until Q_b converges.

Heuristics to generate a feasible solution by adjusting q_1 is presented as follows.

Algorithm B. The algorithm for constructing a feasible solution

Step 1. Recall an infeasible solution.

Step 2. If $g_v(\lambda_v) < 0$, then constraint (5) is satisfied and set $Q_v = q_1$. Otherwise, get $Q_v = \frac{W_v}{\alpha^{n-1}}$.

Step 3. If $g_b(\lambda_b) < 0$, then constraint (6) is satisfied and set $Q_b = q_1$. Otherwise, get $Q_b = \frac{W_b - z_1\sigma\sqrt{pq_1+b}}{\alpha^{n-1}}$.

Step 4. Set $Q^* = \min[Q_v, Q_b]$.

The subgradient method is applied to searching optimal λ_v and λ_b. The subgradient of d and step size s can be computed as follows:

$$d = |g_v(\lambda_v)| + |g_b(\lambda_b)|, \quad s = Q\frac{TC^* - LB}{d^2}$$

where TC^* is the total cost calculated by (4) and the best known feasible solution to (P), and LB is lower bound calculated by (10), and θ is a scalar selected from 0 to 2.

4. **Numerical Example and Result.** To illustrate the performance of the proposed method and sensitivity analysis for space limit of vendor and buyer, the numerical example proposed by Glock [4] is used in Table 1. Sensitivity analysis only for some parameters such as F, b, h_b is done in [4] because space limit is not considered. In this chapter, sensitivity analysis for space limit is done.

As W_b, W_v are decreased from 430 to 150, 410 to 135 by 20, respectively, the trend of decision variables α, q_1, z_1, n are shown in Figure 1. Transportation cost for buyer per shipment (F) is assumed to be \$15, \$25, and \$35. In all cases, optimal solutions are found because TC is equal to LB. As space limit is decreased, TC is naturally increased.

The results are summarized as follows.

As W_v and W_b are decreased,

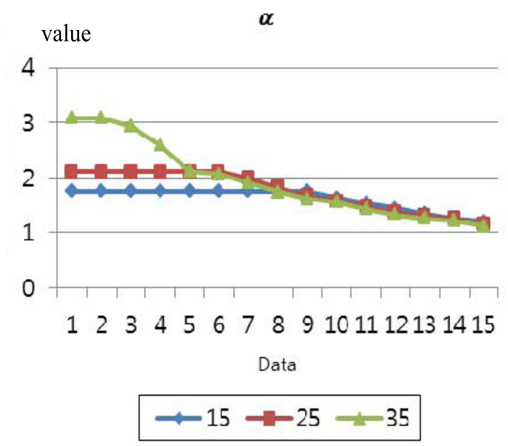
- α is also decreased in all cases because α decides increase rate of shipping batch size. Decrease rate is high in $F = 35$. If transportation cost is high, the shipping batch size is large to reduce the number of shipment.

Table 1. Data for numerical example

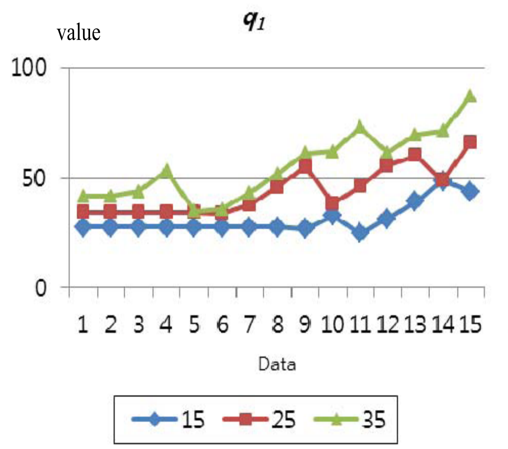
Notation	Value	Notation	Value
D	1000	A_v	\$400
p	1/3200	A_b	\$50
σ	5	h_v	\$4
F	\$25	h_b	\$5
π	\$100	b	0.01

Table 2. The values of α, q_1, z_1, n, TC if W_b is equal to 350 and W_v is changed from 335 to 55 by 20

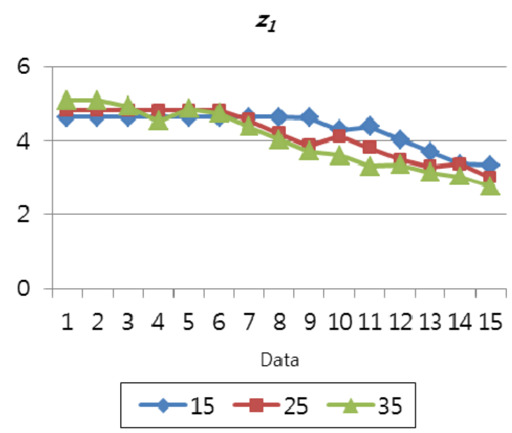
W_v	α	q_1	z_1	n	TC	LB
335	2.11	34.10	4.81	4	1922.37	1922.37
315	2.11	33.62	4.82	4	1922.53	1922.53
295	2.11	31.49	4.77	4	1928.20	1928.20
275	1.98	35.50	4.51	4	1932.93	1932.93
255	1.81	43.06	4.16	4	1936.26	1936.26
235	1.67	30.11	4.37	5	1941.25	1941.25
215	1.54	38.03	4.00	5	1943.40	1943.40
195	1.42	47.48	3.65	5	1948.11	1948.11
175	1.31	59.05	3.31	5	1955.66	1955.66
155	1.25	50.47	3.33	6	1968.77	1968.77
135	1.16	62.98	3.02	6	1979.85	1979.85
115	1.08	78.07	2.75	6	1995.63	1995.63
95	1.00	92.79	2.55	6	2017.07	*2016.94*
75	1.00	72.36	2.62	8	2056.93	*2053.04*
55	1.00	53.97	2.68	10	2141.39	*2140.32*



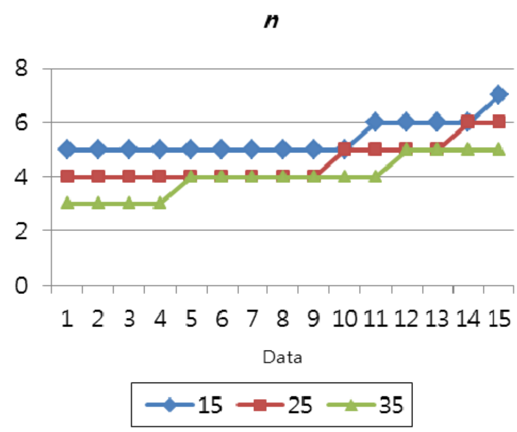
(a) Sensitivity results of α



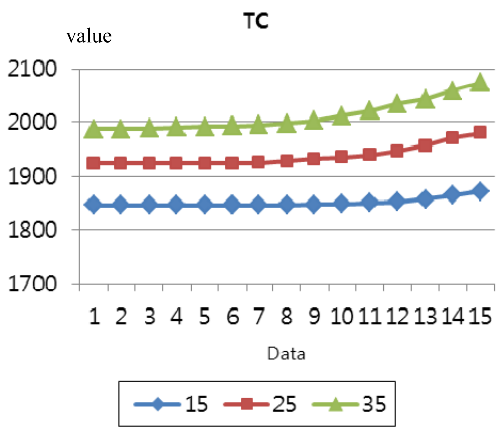
(b) Sensitivity results of q_1



(c) Sensitivity results of z_1



(d) Sensitivity results of n



(e) Sensitivity results of TC

FIGURE 1. Sensitivity analysis for α, q_1, z_1, n, TC if W_b and W_v are changed from 430 to 150 and from 415 to 135 by 20

- n is increased, but z_1 is decreased. When space limit is low, the batch size is small and the frequency of shipment is high. Furthermore, z_1 is small so that safety stock level is low.

- q_1 is increased. To understand this, the relationship among α, q_1, z_1, n should be analyzed. The trend of q_1 is increased, but q_1 sometimes drops at the moment that n rises. For example, q_1 drops at data 5 and 12 for $F = 35$, n rises from 3 to 4, and from 4 to 5 at this moment. Even though q_1 is increased, α is decreased and the final shipping batch (nth batch) size is consistently decreased.

The result of other sensitivity analysis is shown in Table 2 when W_v is decreased from 335 to 55 by 20 and W_b is equal to 350. As W_v is decreased and $W_b = 350$,

- the results are similar to those of the previous sensitivity analysis.
- the optimal solution is not found for $W_v = 95$, 75, and 55 because TC are different from LB. α is equal to 1 and the size of shipping batch is unchanged. Space limit for vendor (W_v) is so small that the size of shipping batch cannot be increased.

5. **Conclusion.** Production-inventory cooperation model with one buyer and one vendor under space limit is studied in this research. We define the constraints for these space limits, and develop the solution method based on Lagrangian relaxation. Moreover, sensitivity analysis for these space limit is conducted to find out the relationship between space limit and decision variables such as α, q_1, z_1, n. Some realistic constraints such as quantity discount and other resource limits can be added in this model, and the procedure for this problem is required to be developed.

REFERENCES

[1] H. K. Alfares and A. M. Ghaithan, Inventory and pricing model with price-dependent demand, time-varying holding cost, and quantity discounts, *Computers & IE*, vol.94, pp.170-177, 2016.

[2] M. Ben-Daya and M. Hariga, Integrated single vendor single buyer model with stochastic demand and variable lead time, *Int. J. Prod. Eco.*, vol.92, pp.75-80, 2004.

[3] B. Ghalebsaz-Jeddi, B. C. Shultes and R. A. Haji, Multi-product continuous review inventory system with stochastic demand, backorders, and a budget constraint, *European Journal of Operational Research*, vol.158, pp.456-469, 2004.

[4] C. H. Glock, A comment: Integrated single vendor-single buyer model with stochastic demand and variable lead time, *Int. J. of Production Economics*, vol.122, pp.790-792, 2009.

[5] S. K. Goyal, A one-vendor multi-buyer integrated inventory model: A Comment, *Eur. J. of Operational Research*, vol.82, pp.209-210, 1995.

[6] M. Hariga, A single-item continuous review inventory problem with space restriction, *Int. J. Prod. Eco.*, vol.128, pp.153-158, 2010.

[7] R. M. Hill, The single-vendor single-buyer integrated production-inventory model with a generalised policy, *Eur. J. of Operational Research*, vol.97, pp.493-499, 1997.

[8] T. Y. Wang and J. M. Hu, An inventory control system for products with optional components under service level and budget constraints, *Euro. J. of Oper. Res.*, vol.189, pp.41-58, 2008.

[9] T.Y. Wang and J. M. Hu, Heuristic method on solving an inventory model for products with optional components under stochastic payment and budget constraints, *Exp. sys. with App.*, vol.37, pp.2588-2598, 2010.

[10] X. Zhao, F. Fan, X. Liu and J. Xie, Storage-space capacitated inventory system with (r, Q) policies, *Operations Research*, vol.55, no.5, pp.854-865, 2007.

[11] S. Tamjidzad and H. Mirmohammadi, An optimal (r,Q) policy in a stochastic inventory system with all-units quantity discount and limited sharable resource, *Eur. J. of Ope. Res.*, vol.247, pp.93-100, 2015.