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IMPLEMENTING THE LIFETIME PERFORMANCE INDEX OF WEIBULL PRODUCTS BASED ON *K*-SAMPLE TYPE II RIGHT CENSORED DATA

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Abstract. *In this article, we propose a maximum likelihood estimator of lifetime performance index under Weibull distribution based on the least squares method and maximum likelihood method under the K-sample type II right censored data. Then the estimator of the lifetime performance index is also utilized to develop the hypothesis testing procedure in the condition of knowing lower specification limit. Finally, we give one practical example to illustrate the testing procedure for determining whether the process is capable.*
Keywords: Process capability analysis, K-sample type II right censored data, Weibull distribution, Maximum likelihood estimator, Testing procedure

1. **Introduction.** In the manufacturing industry, process capability indices are utilized to assess whether product quality meets the required level in conventional automotive, semiconductor, and integrated circuit (IC) manufacturing industries. For instance, Montgomery [1] proposed the process capability index C_L to evaluate the lifetime performance of electronic components, since the lifetime of electronic components exhibits the larger-the-better quality characteristic of time orientation. In addition, for life testing experiments, the experimenter may not always be in a position to observe the life times of all products put on test. This may be because of time limitation and/or other restrictions (such as money, material resources, artificial negligence of typist or recorder, mechanical or experimental difficulties) on data collection. Therefore, censored samples may arise in practice. For the type-II censoring scheme, many researchers [2-6] gave a computational testing procedure based on the maximum likelihood estimator, uniformly minimum variance unbiased estimator, best linear unbiased estimator or chi-squared test statistic to assess lifetime performance index of products (or businesses) with the exponential, Rayleigh or pareto distributions, respectively. In many experiments, the life-testing device is restricted such that the test specimens will be grouped and placed on a life test. Several authors including [7,8] have discussed inferential procedures for the scale parameter of exponential distribution based on K-sample type II censoring. Nowadays, the Weibull distribution is a broadly used statistical model in lifetime data analysis. In this paper, we consider that the lifetime of products may be modeled by a Weibull distribution and the K-sample type II right censored data will be applied in assessing lifetime performance index of products. We will apply the data transformation technology to construct the maximum likelihood estimator of C_{L_Y} based on the least squares method (See Section 2 and Section 5) and the maximum likelihood method. The maximum likelihood estimator of C_{L_Y} is then used to develop the hypothesis testing procedure. The hypothesis testing

procedure can be employed by managers and purchasers to determine whether the lifetime performance meets the required level.

The rest of this paper is organized as follows. In Section 2, we discuss the relationship between the lifetime performance index and the conforming rate of products. In Section 3, we present the maximum likelihood estimator of C_{L_Y} and its main properties. In Section 4, we develop the hypothesis testing procedure for the lifetime performance index. One example is illustrated to employ the hypothesis testing procedure in Section 5. Finally, some concluding comments are made in Section 6.

2. **The Lifetime Performance Index and the Conforming Rate.** Suppose that the lifetime (X) of products has the exponential distribution with the probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) as given by

$$f_X(x) = \frac{\beta x^{\beta-1}}{\eta^\beta} \exp\left(-\left(\frac{x}{\eta}\right)^\beta\right), \ x > 0,\ \eta > 0,\ \beta > 0, \tag{1}$$

and

$$F_X(x) = 1 - \exp\left(-\left(\frac{x}{\eta}\right)^\beta\right), \ x > 0,\ \eta > 0,\ \beta > 0, \tag{2}$$

respectively (also see Nelson [9] and Lee et al. [10,11]). Here the shape parameter β is known. If the shape parameter β is unknown, then we use the least squares method based on the minimization of the Residual Sum of Squares (RSS) to select the optimum value of the shape parameter β of the Weibull distribution and β is given. Note that the lifetime of products is a larger-the-better type quality characteristic, a longer lifetime implies a better product quality. Hence, the lifetime is generally required to exceed L unit times so as to be both financially profitable and customer-satisfactory, where L is the lower specification limit. Montgomery [1] developed a capability index C_L to measure the larger-the-better quality characteristics. C_L is defined as follows:

$$C_L = \frac{\mu - L}{\sigma}, \tag{3}$$

where μ denotes the mean lifetime, σ represents the lifetime standard deviation, and L is the lower specification limit.

To assess the lifetime performance of products, C_L can be defined as the lifetime performance index. The lifetime X has a Weibull distribution with p.d.f. as in Equation (1), and then the mean and standard deviation of lifetime of products are given respectively, by

$$\mu_X = E(X) = \eta\Gamma(1 + 1/\beta), \tag{4}$$

$$\sigma_X = \sqrt{Var(X)} = \eta M, \tag{5}$$

where $M = \sqrt{\Gamma(1 + 2/\beta) - [\Gamma(1 + 1/\beta)]^2}$.

Substitute Equations (4)-(5) into Equation (3), and let L_X be the lower specification limit of products, then the lifetime performance index reduces to

$$C_{L_X} = \frac{\mu_X - L_X}{\sigma_X} = \frac{\Gamma(1 + 1/\beta)}{M} - \frac{L_X}{\eta M}, \ -\infty < C_{L_X} < \frac{\Gamma(1 + 1/\beta)}{M}. \tag{6}$$

Since X has a Weibull distribution, by the transformation $Y = X^\beta$, $\beta > 0$, then Y is distributed as exponential with the mean $\theta = \eta^\beta$. The p.d.f. of Y is

$$f_Y(y) = \frac{1}{\theta} \exp\left(-\frac{y}{\theta}\right), \ y > 0,\ \theta > 0. \tag{7}$$

Moreover, there are several important properties as follows.

- Let L_Y be the lower specification limit of exponential lifetime model, and the lifetime performance index is given by

$$C_{L_Y} = \frac{\mu_Y - L_Y}{\sigma_Y} = \frac{\theta - L_Y}{\theta} = 1 - \frac{L_Y}{\theta},\ C_{L_Y} < 1, \tag{8}$$

where $L_Y = (L_X)^\beta$ and $\theta = \eta^\beta$.

- The c.d.f. of Y is

$$F_Y(y) = 1 - \exp\left(-\frac{y}{\theta}\right),\ y > 0, \tag{9}$$

where $\theta = \eta^\beta$, $\theta > 0$.

- From Equations (6)-(8), we have

$$C_{L_X} = \frac{\Gamma(1 + 1/\beta)}{M} - \frac{(1 - C_{L_Y})^{\frac{1}{\beta}}}{M}. \tag{10}$$

Thus, a strictly increasing relationship exists between C_{L_X} and C_{L_Y}. Furthermore, if the lifetime of a product exceeds the lower specification limit (i.e., $X \geq L_X$), then the product is labeled as a conforming product. The ratio of conforming products is known as the conforming rate, and can be defined as

$$P_r = P(X \geq L_X) = 1 - F(L_X) = \exp\left(-\left(\frac{L_X}{\eta}\right)^\beta\right). \tag{11}$$

By the transformation $Y = X^\beta$ and the lower specification limit $L_Y = (L_X)^\beta$, Equation (11) can be rewritten as

$$P_r = P(X \geq L_X) = P(Y \geq L_Y) = 1 - F_Y(L_Y) = \exp\left(-\frac{L_Y}{\theta}\right) = \exp(C_{L_Y} - 1). \tag{12}$$

Obviously, a strictly increasing relationship exists between conforming rate (P_r) and the lifetime performance index (C_{L_Y}). In addition, the relationship of various C_{L_Y} values and the corresponding conforming rates (P_r) is presented in Table 1 of Hong et al. [2].

Table 1. Six-sample type II right censored data

Group 1	0.31	0.66	1.54	1.70	1.82	1.89	2.17	2.24	-	-
Group 2	0.00	0.18	0.55	0.66	0.71	1.30	1.63	2.17	-	-
Group 3	0.49	0.64	0.82	0.93	1.08	1.99	2.06	2.15	-	-
Group 4	0.02	0.06	0.50	0.70	1.17	2.80	3.57	3.72	-	-
Group 5	0.20	0.78	0.80	1.08	1.13	2.44	3.17	5.55	-	-
Group 6	1.34	1.49	1.56	2.10	2.12	3.83	3.97	5.13	-	-

3. **MLE of Lifetime Performance Index.** In this section, we consider the case of the K-sample type II right censoring and the lifetime (X) of a product follows the Weibull distribution with p.d.f. as Equation (1). We denote the total number of observations in the ith sample by n_i ($i = 1, 2, \ldots, K$), and the type II right censored sample observed from the ith sample by $X_{i(1)} \leq X_{i(2)} \leq \cdots \leq X_{i(r_i)}$, $1 \leq r_i \leq n_i$, $i = 1, 2, \ldots, K$. When β is known (or given), convert all $X_{i(j)}$ into $Y_{i(j)}$ through the equation $Y = X^\beta$. Then $Y_{i(1)} \leq Y_{i(2)} \leq \cdots \leq Y_{i(r_i)}$, $1 \leq r_i \leq n_i$, $i = 1, 2, \ldots, K$ are K independent type II right

censored samples from the exponential distribution with p.d.f. as Equation (7). Therefore, we can show that the maximum likelihood estimator $\hat{\theta}$ of θ is given by

$$\hat{\theta} = \frac{\sum_{i=1}^{K}\left(\sum_{j=1}^{r_i} Y_{i(j)} + (n_i - r_i)Y_{i(r_i)}\right)}{\sum_{i=1}^{K} r_i} = \frac{\sum_{i=1}^{K} W_i}{r}, \tag{13}$$

where $W_i = \sum_{j=1}^{r_i} Y_{i(j)} + (n_i - r_i)Y_{i(r_i)}$, $r_i \leq n_i$, $i = 1, 2, \ldots, K$ and $r = \sum_{i=1}^{K} r_i$. And let $W = \sum_{i=1}^{K} W_i$, we show that $\frac{2W}{\theta} \sim \chi^2_{2r}$ and the Fisher information number $I(\theta) = \frac{r}{\theta^2}$. Hence, we also show that $\hat{\theta} \sim AN\left(\theta,\ \frac{\theta^2}{r}\right)$. According to the invariance of maximum likelihood estimator (see Zehna [12]), the maximum likelihood estimator $\hat{C}_{L_Y}$ of C_{L_Y} is given by $\hat{C}_{L_Y} = 1 - \frac{L_Y}{\hat{\theta}} = 1 - \frac{rL_Y}{W}$. Furthermore, according to the Delta Method (see Casella and Berger [13]), we show that

$$\hat{C}_{L_Y} \sim AN\left(C_{L_Y},\ \frac{(1 - C_{L_Y})^2}{r}\right). \tag{14}$$

4. **Testing Procedure for the Lifetime Performance Index.** In this section, a hypothesis testing procedure will be constructed to assess whether the lifetime performance index adheres to the required level. As we noted earlier, the data transformation technology enables the calculation of important properties to be easy. From Equation (12), the performance index C_{L_Y} is available to determine the conforming rate of products. To assess the lifetime performance of products, constructing a testing procedure for the index C_{L_Y} is feasible. Assuming the required index value of performance is larger than c, the null hypothesis H_0: $C_{L_Y} \leq c$ and the alternative hypothesis H_1: $C_{L_Y} > c$ are constructed, where c denotes the target value. By using $\hat{C}_{L_Y}$ as the test statistic, the rejection region can be expressed as $\{\hat{C}_{L_Y} > C_0\}$. Given a specified significance level α, the critical value C_0 can be derived:

$$C_0 = 1 - \frac{2r(1-c)}{CHIINV(1-\alpha,\ 2r)}, \tag{15}$$

where c, α and r denote the target value, the specified significance level and the total number of observed failures, respectively. Moreover, we also find that C_0 is independent of K, n_i and r_i, $i = 1, 2, \ldots, K$. Alternatively, we can also use asymptotic theory for large sample; the critical value C_0 can be derived:

$$C_0 = c + Z_{1-\alpha} \times \left(\frac{1-c}{\sqrt{r}}\right). \tag{16}$$

5. **One Practical Example.** Here, we use one practical example to illustrate the hypothesis testing procedure proposed based on the insulating fluid failure data of Nelson [9] which gives 60 times to breakdown in minutes of an insulating fluid subject to high voltage stress.

Example 5.1. *The six-sample type II right censored data, in Table 1, are taken from Nelson [9] who gives the failure times observed in the form of six groups each with ten insulating fluids subject to high voltage stress. And Nelson only said that the data of the failure times of six groups may have the Weibull distribution.*

Then, we state the proposed testing algorithmic procedure about C_{L_Y} as follows.

Step 1: Let X have a Weibull distribution with c.d.f. as Equation (2), and then $F_X(x)$ satisfies $-\ln(1-F_X(x)) = \lambda x^\beta$, $x > 0$, $\lambda \left(= \frac{1}{\theta} = \frac{1}{\eta^\beta}\right) > 0$, $\beta > 0$. We combine all the type II censored sample into one sample and $t_1 \le t_2 \le \cdots \le t_r$, $r = \sum_{i=1}^{6} r_i = 48$ are the corresponding combined sample for the six-sample type II censored data in Table 1. Let the mean estimate of $F(t_i)$ be $i/(r+1)$, $i = 1, 2, \ldots, 48$ by using the approximated equation $-\ln(1 - \frac{i}{r+1}) \approx \lambda t_i^\beta$, $i = 1, 2, \ldots, 48$ and the least squares method based on the minimization of the RSS, that is, for each $\beta = 0.1(0.1)1.6$, $\min_\lambda \sum_{i=1}^{48} \left[-\ln\left(1 - \frac{i}{48+1}\right) - \lambda t_i^\beta\right]^2 (= \text{RSS})$. Hence, we obtain that $\beta = 1.2$ is very close to the optimum value and $\hat{\lambda} = 0.492985$. Then, β is defined as known.

Step 2: Take the data transformation of $y_{i(j)} = (x_{i(j)})^{1.2}$, $j = 1, 2, \ldots, 8$, $i = 1, 2, \ldots, 6$, where $x_{i(j)}$ is the observed failure time in Table 1. And $r = 48$.

Step 3: The lower lifetime limit L_Y is assumed to be 0.2028. That is, the electrical insulating fluid is defined as a conforming product if the lifetime of an electrical insulating fluid exceeds 0.2646 minute. To deal with concerns about the lifetime performance, the conforming rate P_r of products is required to exceed 81 percent. Referring to Table 1 of Hong et al. [2], the C_{L_Y} value is required to exceed 0.80. Thus, the target value is set at $c = 0.8$. The testing hypothesis H_0: $C_{L_Y} \le 0.8$ v.s. H_1: $C_{L_Y} > 0.8$ is constructed.

Step 4: Specify a significance level $\alpha = 0.05$ and obtain the critical value $C_0 = 0.8398$ by Equation (15), according to $c = 0.8$ and $r = 48$.

Step 5: Calculate the value of test statistic $\hat{C}_{L_Y} = 0.9343$.

Step 6: Because $\hat{C}_{L_Y} = 0.9343 > C_0 = 0.8398$, the null hypothesis is rejected. We can conclude that the lifetime performance index of electrical insulating fluid operation meets the required level.

Alternatively, if we apply asymptotic normal distribution to this case, the critical value $C_0 = 0.8566$ is obtained by Equation (16). Because $\hat{C}_{L_Y} = 0.9343 > 0.8566$, the null hypothesis is still rejected.

6. **Concluding Remarks.** In practice, lifetime performance index C_{L_Y} is used to measure the potential and performance of a process, since the lifetime of products exhibits the larger-the-better quality characteristic. Moreover, we apply the data transformation technology to construct the maximum likelihood estimator of C_{L_Y} under Weibull distribution based on K-sample type II right censored data. Our estimation method (including the least squares method and the maximum likelihood method) is simple and does not require a numerical method to solve the parameter of the Weibull distribution for the K-sample type II right censored data. This is an advantage that it is different from other estimation methods. The maximum likelihood estimator of C_{L_Y} is then used to develop the hypothesis testing procedure. In addition, our proposed testing algorithmic procedure is easily applied to evaluate whether the lifetime of products meets requirements. In future research on this problem, it would be interesting to deal with the Weibull products based on progressive type II censored K-samples.

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