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ECONOMIC ANALYSIS ABOUT RELIABILITY ON PARALLEL SYSTEM UNDER MULTI CONSTRAINTS

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Abstract. *When decision maker wants to improve the reliability of the system, they will face the cost and volume of increase, etc. So how to find the optimal combination scheme with limited resources becomes a big problem. The paper uses indifference curve analysis, symmetry and Kuhn-Tucker condition and non-linear goal programming to find the best solution under multi constraints, and try to classify the running phase of the system, in order to observe better for the marginal economic benefits brought by each stage.*

Keywords: Reliability, Parallel system, Exponential distribution, Double constraints

1. Introduction.

1.1. **Reliability introduction.** Reliability theory was a newly emerging interdisciplinary subject in 1960s and analyzes the probability of random events which characterize the specified function of product. It is established on the basis of probability theory, which is an area of study focused on machine maintenance [1]. With the development of reliability theory, it gradually needs much frontier knowledge and tools in mathematics, while reliability mathematics has laid a good foundation for it. In practical reliability problems, the mathematics used can be divided into two categories: probability model and statistical model. Probability model infers the reliability indices of system on the basis of system structure and life distribution of components; while statistical model evaluates and tests the life of components or system on the basis of observed data. In the paper, statistical model is applied. Currently, the main researches focus on the reliability indices of system and optimal detecting time which is determined by reliability indices to avoid the occurrence of faults and reduce the losses caused by faults, such as [2-4]. For the parallel system with n element which obeys exponential distribution, Xie et al. have made some analyses of reliability about extreme value [5]. When the economic constraint condition is not considered, we get that when the failure rates of all components are equal, the system reliability can reach the minimum value, also under the performance and costs constraints, the result is same as only performance constraints, but it has least costs to support its feasible solution space in order to avoid becoming empty set. Therefore, in practice, it is better to choose a relatively poor one from a good parallel product than select products with the same quality, because the reliability of the former is better than that of the latter. In the process of running, according to the actual situation of system, if the curve of failure rate is above the envelope curve, then all the components of the system have to be replaced; if the curve of failure rate is still relatively distant from the envelope curve, then the components with the highest failure rate need to be replaced according to the economy principle. However, in that paper, we only consider the lower envelope and the unite flexibility of failure rate. Scholars research rarely on the double constraints like under performance and economic condition in the domestic and abroad; research which has done basically is qualitative analysis or uses numerical analysis to find the approximation value of solution in actual system [6-8]. Ida et al. proposed a genetic algorithm for solving reliability problems which belongs to intelligent algorithm [9-12]. So all their researches were focusing on engineering method, but this paper uses operational research and economic optimization method to find the optimal solution which is exact solution. Lastly, we use matrix to describe multi constraints of the system. The paper uses indifference curve analysis, symmetry and Kuhn-Tucker condition and non-linear goal programming to find the maximum value distribution [13,14], so that we can choose the best solution in design period, and when we should think about to update system.

1.2. **The definition of the main indicators of reliability.**

(1) Reliability

The definition of reliability $R(t)$: it is the probability that product completes the required function under the specified conditions and within the prescribed time.

If the life distribution of product is $F(t)$, $t > 0$, the reliability

$$R(t) = P(T \geq t) = 1 - F(t) \tag{1}$$

This is a function of time (t), so it can be called as reliability function. To the components obeying exponential distribution λ, its reliability is $e^{-\lambda t}$, $t \geq 0$.

(2) Failure rate

Failure rate $\lambda(t)$: it is the probability of occurring failure in the unit of time after product has worked a period of time (t). According to reliability theory,

$$\lambda(t) = \frac{f(t)}{1 - F(t)} \tag{2}$$

When $t > 0$, the failure rate of exponential distribution is constant λ.

(3) System parameter specification

A: represents normal working events of system.

A_i: represents normal working events of the element i.

λ_i: represents failure rate of the element i.

$R_s(t)$: represents system reliability, that is, $P(A) = R_s$.

$R_i(t)$: represents reliability of the element i, that is, $P(A_i) = R_i$.

Parallel system: it is a system consisting of n components. As long as one of these elements works, the system can work; only when all the units fail, the system would fail.

According to the property of probability, the normal working probability of system $P(A) = P\left(\bigcup_{i=1}^{n} A_i\right)$ is as follows:

$$R_s(t) = 1 - \prod_{i=1}^{n} (1 - R_i(t)) = 1 - \prod_{i=1}^{n} \left(1 - e^{-\lambda_i t}\right) \tag{3}$$

2. **Model Analysis.**

2.1. **Maximum reliability analysis under two constraints.**

When $t = 1$, it can be transformed to the following model to obtain the maximum value of the original model

$$\min R = \prod_{i=1}^{n} (1 - e^{-\lambda_i}) \\ \text{s.t.} \begin{cases} \sum_{i=1}^{n} \lambda_i \leq c \\ \sum_{i=1}^{n} f(\lambda_i) \leq u \\ \lambda_i > 0 \end{cases} \tag{4}$$

and model is equivalent to

$$\min R^* = \sum_{i=1}^{n} \text{In} \left(1 - e^{-\lambda_i}\right) \\ \text{s.t.} \begin{cases} \sum_{i=1}^{n} \lambda_i \leq c \\ \sum_{i=1}^{n} f(\lambda_i) \leq u \\ \lambda_i > 0 \end{cases} \tag{5}$$

Set

$$\begin{aligned} g_1(\lambda_1, \lambda_2, \cdots, \lambda_n) &= -\sum_{i=1}^{n} \lambda_i + c \\ g_2(\lambda_1, \lambda_2, \cdots, \lambda_n) &= -\sum_{i=1}^{n} f(\lambda_i) + u \end{aligned} \tag{6}$$

The constraints of the original model become:

$$\begin{aligned} g_1(\lambda_1, \lambda_2, \cdots, \lambda_n) &= -\sum_{i=1}^{n} \lambda_i + c \geq 0 \\ g_2(\lambda_1, \lambda_2, \cdots, \lambda_n) &= -\sum_{i=1}^{n} f(\lambda_i) + u \geq 0 \end{aligned} \tag{7}$$

According to well-known Kuhn-Tucker conditions, optimal value must satisfy the following conditions:

$$\begin{cases} \nabla R^*(\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*) - \kappa \nabla g_1(\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*) - \gamma \nabla g_2(\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*) = 0 \\ \kappa g_1(\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*) = 0 \\ \gamma g_2(\lambda_1^*, \lambda_2^*, \cdots, \lambda_n^*) = 0 \\ \lambda_i^* > 0, \ \kappa > 0, \ \gamma > 0 \end{cases}$$

Expanding the equation we obtain the following:

$$\begin{cases} \left(\frac{1}{e^{\lambda_1^*}-1}, \frac{1}{e^{\lambda_2^*}-1}, \cdots, \frac{1}{e^{\lambda_n^*}-1}\right) + \kappa(1, 1, \cdots, 1) - \gamma\left(f'(\lambda_1^*), f'(\lambda_2^*), \cdots, f'(\lambda_n^*)\right) = 0 \\ \kappa\left(-\sum_{i=1}^{n} \lambda_i^* + c\right) = 0 \\ \gamma\left(-f'(\lambda_i^*) + u\right) = 0 \\ \lambda_i^* > 0, \ \kappa > 0, \ \gamma > 0 \end{cases} \tag{8}$$

The equation can be divided into the following circumstances:

i) $\kappa = 0$, $\gamma = 0$, original equation has no solution.

ii) $\kappa \neq 0$, $\gamma = 0$, original equation has no solution.

iii) $\kappa = 0$, $\gamma \neq 0$, get the results as follows:

$$\begin{cases} \frac{1}{f'(\lambda_1^*) * \left(e^{\lambda_1^*}-1\right)} = \frac{1}{f'(\lambda_2^*)\left(e^{\lambda_2^*}-1\right)} = \cdots = \frac{1}{f'(\lambda_n^*) * \left(e^{\lambda_n^*}-1\right)} = \gamma \\ -\sum_{i=1}^{n} f(\lambda_i^*) + u = 0 \\ \lambda_i^* > 0, \quad \gamma > 0 \end{cases} \tag{9}$$

iv) $\kappa \neq 0$, $\gamma \neq 0$, get the results as follows:

$$\begin{cases} \frac{1}{\left(e^{\lambda_i^*}-1\right)} + \kappa - \gamma f'\left(\lambda_i^*\right) = 0 \\ -\sum\limits_{i=1}^{n} \lambda_i^* + c = 0 \\ -\sum\limits_{i=1}^{n} f\left(\lambda_i^*\right) + u = 0 \\ \lambda_i^* > 0,\ \kappa > 0,\ \gamma > 0,\ i = 1, 2, \cdots, n \end{cases} \tag{10}$$

when $t \neq 1$, set $e^t = a$, due to $t > 0$, $a > 0$. The results can be replaced by substituting e to a. For the i-th element, the amount of decreasing failure rate aroused by increasing one unit of cost is $\frac{1}{f'(\lambda_i)}$, as well as the amount of increasing reliability aroused by decreasing one unit of failure rate is $\frac{1}{e^{\lambda_i}-1}$, it can be seen from Formula (9), if optimum point is not on the boundary constraint of performance, to reach the maximum reliability, all the reliability increasing amounts arising from element of unit costs should be equal, and the conclusion is similar with consumer equilibrium in microeconomics. The paper calls it costs-failure rate equilibrium, and the ratio is called the marginal reliability of unit cost. On the other hand, when optimum point on the boundary constraint of performance, we cannot just consider the element reliability increases by adding a unit cost, but also consider the element performance reducing by adding a unit cost, while other parts must complement the reducing performance which was reduced by the element, it can be seen from Formula (10), to reach maximum reliability, marginal reliability is corrected as $\gamma f\left(\lambda_i^*\right) - \kappa$, and the modified equilibrium is called boundary-costs-failure rate equilibrium.

Since $\lambda^* = \left(\frac{n}{u}, \frac{n}{u}, \cdots, \frac{n}{u}\right)$ satisfies Formula (9), it is one of K-T points, and we get a lower bound of the optimal solution as follows:

$$R_{\inf} = 1 - \left(1 - e^{-\frac{u}{n}t}\right)^n \tag{11}$$

However, whether the K-T point is optimal solutions, now we use indifference curve to analyze and divide $g_2(\lambda)$ into three cases depending on the marginal rate of substitution. Take $n = 2$ for example, as shown in Figure 1, when $\frac{d^2\lambda_2}{d\lambda_1^2}_{g_2} > \frac{d^2\lambda_2}{d\lambda_1^2}_{R_s}$, the system has two best solutions, and the solutions are symmetry on the basis of $y = x$, especially the best solution is boundary of feasible solution space, corresponding to K-T points of Formula (8), when having n variable, according to symmetry the best solution is $\lambda^* = \left(\frac{n}{u}, \frac{n}{u}, \cdots, \frac{n}{u}\right)$, and maximum reliability is Formula (11), the whole economic constraints curve boundary is best solution, this is seldom case in reality. It is easy to popularize $n > 2$ because of symmetry.

When $\frac{d^2\lambda_2}{d\lambda_1^2}_{g_2} < \frac{d^2\lambda_2}{d\lambda_1^2}_{R_s}$, the system has only one best solution, corresponding to K-T points of Formula (10) as shown in Figure 2. However, anyway in reality $R_{\inf} = 1 - \left(1 - e^{-\frac{u}{n}t}\right)^n$ is an acceptable solution.

So the system of upper and lower envelope is as shown in Figure 3.

Because system reliability is always in shadow during the process of running, upper envelope is maximum reliability curve, and lower envelope is minimum, and the middle line is the lower bound of maximum envelope got by Formula (11). In this paper, the system is divided into four periods, as shown in Figure 3. In breaking-in period, the difference between maximum and minimum reliability is not significant, but after running for a period of time, reliability difference comes out in parallel system, as part of running, the slowing down speed of system reliability gradually shows the advantage with minimum reliability, and keeps quite a long period of time also including the recession; after entering the period of scrap, the envelopes are gradually approaching; no matter what kind of scheme selection, all elements should be replaced and upgraded again. In running process, if the failure

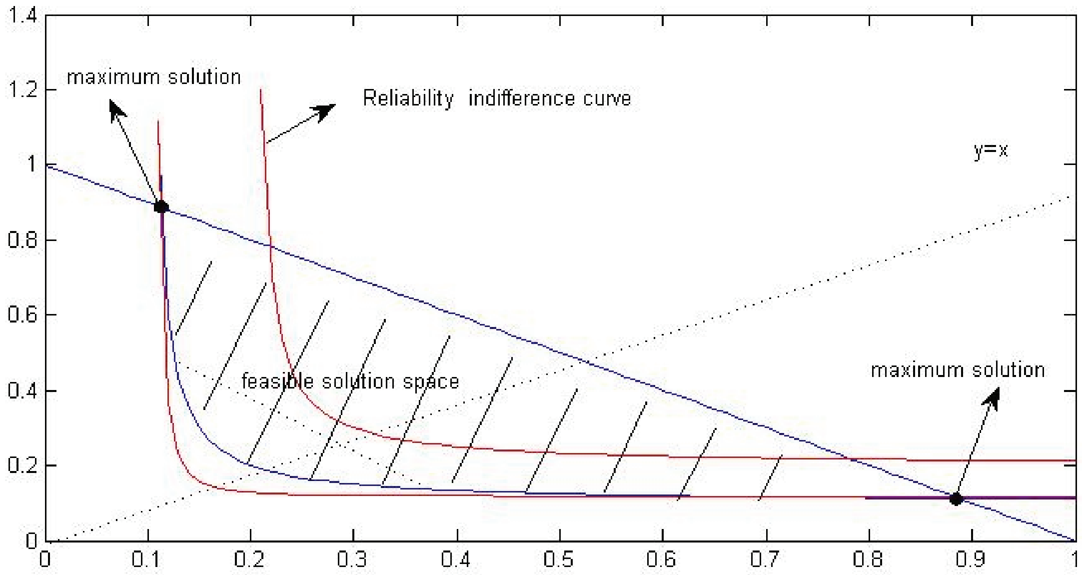

FIGURE 1. The first case of maximum indifference analysis

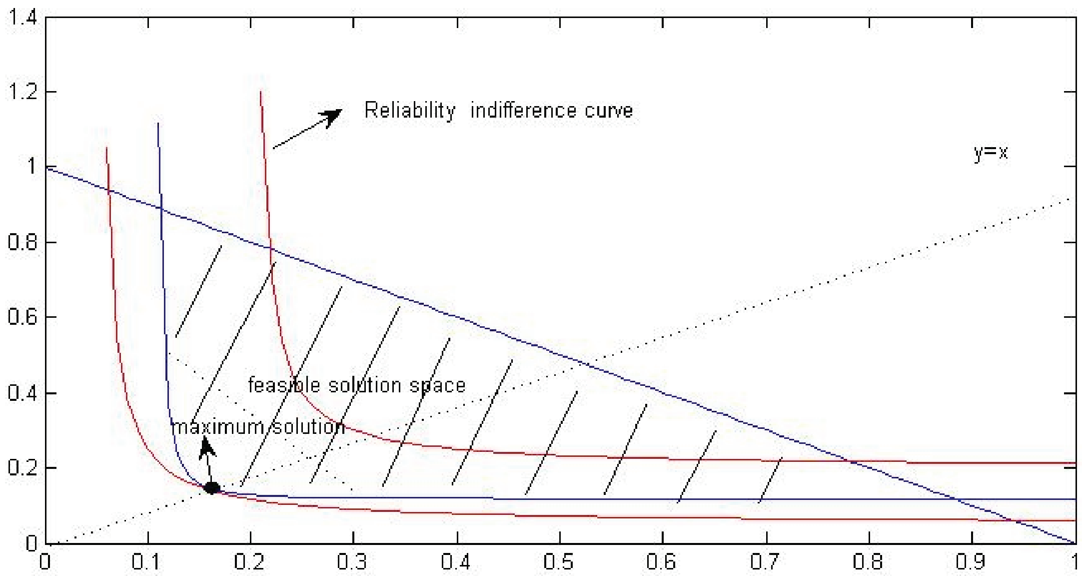

FIGURE 2. The second case of maximum indifference analysis

rate is between $R_{\inf}$ and c, it can be called ideal region, and the failure rate probability will become very small when running on the ideal area; it is the most stable state.

2.2. **Reliability analysis under multi constraints.** Now we consider the system not only contains two constraints like performance, economic, weight and capacity constraints. Assuming that the system has m index, it can be described by the vector:

$$\begin{pmatrix} f_1 \\ f_2 \\ \cdots \\ f_m \end{pmatrix}$$

Every index is function of $\lambda_1, \lambda_2, \cdots, \lambda_n$; in addition, introduce positive and negative deviation variables, denoted as d^+, d^-; positive deviation variables represent the decision value which exceeds the target value of the part and negative deviation variables represent the decision value does not reach the target value of the part. Because the decision

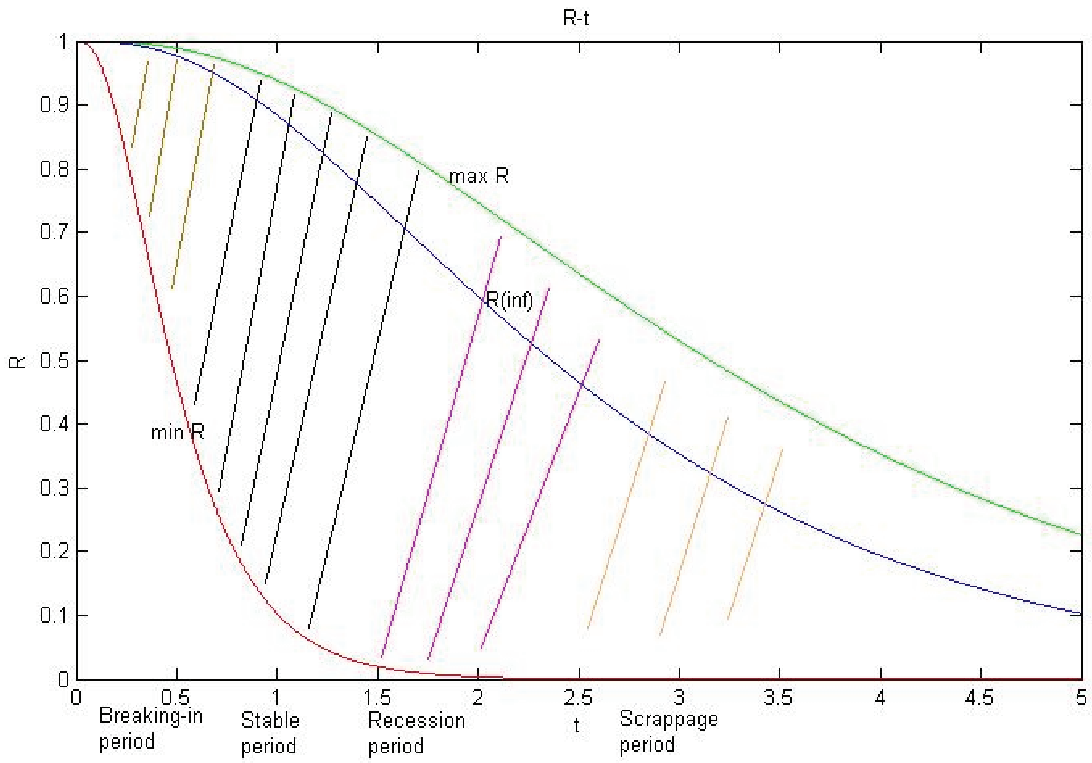

FIGURE 3. Upper and lower envelope

value could not exceed the target value and also not reach the target value, that is the constant $d^+ \times d^- = 0$.

When decision makers require to achieve these goals, they have primary and secondary order of priority, the first goal requirement given priority factor P_1, a secondary goal given priority factor $P_2, \ldots$, and sets $P_k >> P_{k+1}$, $k = 1, 2, \ldots, K$. It indicates P_k has a bigger priority than P_{k+1}. That first of all, it ensures the achievement of objectives P_1, then we can not consider the secondary target $P_2, \ldots, P_2$ is based on the achievement P_1, and so on.

Now we consider the objective function in the target planning, when every goal determined, decision makers must limit the deviation from the target, therefore the objective function of goal programming is $\min z = g(d^+, d^-)$. There are three kinds of basic forms:

(1) Just reach the target value requirements, that is, positive and negative deviation variables should be as small as possible:

$$\min z = g(d^+ + d^-)$$

(2) Not exceeding the target value requirements, that is allowing not reach the target value, positive deviation variables should be as small as possible:

$$\min z = g(d^+)$$

(3) Exceeding the target value requirements, it means that exceeding the amount is not restricted, but negative deviation variables should be as small as possible:

$$\min z = g(d^-)$$

So the model can be described as:

$$\min z = \sum_{l=1}^{L} P_l \sum_{k=1}^{K} \left(w_{lk}^- d_k^- + w_{lk}^+ d_k^+ \right)$$

$$\text{s.t.}\ \begin{cases} f_i\left(\lambda_1,\lambda_2,\cdots,\lambda_n\right)+d_i^- - d_i^+ = b_i, \quad i=1,2,\cdots,m \\ \lambda_i > 0, \quad i=1,2,\cdots,m \\ d_i^-,\ d_i^+ \geq 0,\ i=1,2,\cdots,m \\ K+L=m \end{cases} \tag{12}$$

In the formula, w_{lk}^-, w_{lk}^+ represent coefficients of the same level. For hardware system, the focus is on system reliability, so if f_1 represents reliability goal, the priority factor is P_1, the goal only contains d_1^-, and the economic and performance constraints are hard conditions. For other goals we consider it is the same level. Then the model is simplified as:

$$\min z = P_1 d_1^- + P_2\left(\sum_{k=3}^{m}\left(d_k^- + d_k^+\right)\right)$$

$$\text{s.t.}\ \begin{cases} f_i\left(\lambda_1,\lambda_2,\cdots,\lambda_n\right)+d_i^- - d_i^+ = b_i, \quad i=1,4,\cdots,m \\ \sum\limits_{i=1}^{n}\lambda_i \leq c \\ \sum\limits_{i=1}^{n} f(\lambda_i) \leq u \\ \lambda_i > 0,\ i=1,4,\cdots,m \\ d_i^-,\ d_i^+ \geq 0,\ i=1,4,\cdots,m \end{cases} \tag{13}$$

So analyzing like when $\frac{d^2\lambda_2}{d\lambda_1^2}_{g2} < \frac{d^2\lambda_2}{d\lambda_1^2}_{R_s}$, the system has only one best solution, corresponding to K-T points of Formula (9); according to symmetry the best solution is $\lambda^* = \left(\frac{n}{u},\frac{n}{u},\cdots,\frac{n}{u}\right)$, and maximum reliability is Formula (10). Lastly, when $\frac{d^2\lambda_2}{d\lambda_1^2}_{g2} > \frac{d^2\lambda_2}{d\lambda_1^2}_{R_s}$ the whole economic constraints curve boundary is best solution. Especially we consider the second situation, it can transform into the following optimal on the boundary model:

$$\min z = \sum_{k=3}^{m}\left(d_k^- + d_k^+\right)$$

$$\text{s.t.}\ \begin{cases} f_i\left(\lambda_1,\lambda_2,\cdots,\lambda_n\right)+d_i^- - d_i^+ = b_i,\ i=4,\cdots,m \\ \frac{1}{\left(e^{\lambda_i}-1\right)} + \kappa - \gamma f'(\lambda_i) = 0 \\ -\sum\limits_{i=1}^{n}\lambda_i + c = 0 \\ -\sum\limits_{i=1}^{n} f(\lambda_i) + u = 0 \\ \lambda_i^* > 0,\ \kappa > 0,\ \gamma > 0,\ i=1,2,\cdots,n \end{cases} \tag{14}$$

Then the following analysis like front, we can use Kuhn-Tucker condition for further analysis.

3. **Conclusion.** The paper uses indifference curve analysis, symmetry and Kuhn-Tucker condition to analyze reliability of the parallel system with n element, which obeyed exponential distribution under the double constraints, no matter what kind of failure rate flexibility. When the failure rates of all components are equal, the system reliability can reach the minimum value, that is have least expense to provide feasible solution and the solution is the worst scheme. In addition, the paper proposes how to reach the maximum reliability under various circumstances, we use K-T condition to give the results, if optimum point is not on the boundary constraint of performance, to reach the maximum reliability, all the reliability increasing amounts arising from element of unit costs should be equal, and we call it costs-failure rate equilibrium. When optimum point on the boundary constraint of performance, to reach maximum reliability, marginal reliability is corrected as $\gamma f(\lambda_i^*) - \kappa$, the modified equilibrium is called boundary-costs-failure rate equilibrium, and we use indifference analysis to get the best solution which is corresponding

to the two equilibriums. By contrast, the system is divided into four periods: breaking-in period, stable period, recession period and scrap period. In breaking-in period, the difference between maximum and minimum reliability is not significant, but after running for a period of time, reliability difference comes out in parallel system, the slowing down speed of system reliability gradually shows the advantage with minimum reliability, and keeps quite a long period of time also including the recession; after entering the period of scrap, the envelopes are gradually approaching, and no matter what kind of scheme selection, all elements should be replaced and upgraded again. In running process, if the failure rate is between $R_{\inf}$ and $R_{\max}$, it can be called ideal region, and the failure rate probability will become very small when running on the ideal area, it is the most stable state, lastly we use non-linear goal programming to transform the multi constraints into simple model, and the conclusion as same as double constraints.

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