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PERMANENCE AND STABILITY OF A DIFFUSIVE EPIDEMIC MODEL WITH FEEDBACK CONTROLS

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Abstract. *In this paper, a four-dimensional reaction-diffusion system of SI epidemic model with spatial diffusion and feedback controls is proposed. The well-posedness of the diffusive system is established and the permanence of the system is obtained by applying the comparison principle. Then, the global stability of two nonnegative equilibria is investigated by constructing suitable Lyapunov functions respectively. It is shown that the disease-free equilibrium is globally asymptotically stable when the basic reproduction number R_0 is smaller than one and the endemic equilibrium is globally asymptotically stable when R_0 is larger than one. Finally, some numerical simulations are given to illustrate the theoretical results.*

Keywords: SI epidemic model, Reaction-diffusion system, Feedback control, Permanence, Stability

1. **Introduction.** Recently, Chen and Sun [1] have considered the following SI epidemic model with feedback controls:

$$\begin{cases} \frac{\mathrm{d}S(t)}{\mathrm{d}t} = S(t)(r - aS(t) - bI(t) - c_1u_1(t)), \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} = I(t)(bS(t) - \mu - fI(t) - c_2u_2(t)), \\ \frac{\mathrm{d}u_1(t)}{\mathrm{d}t} = -e_1u_1(t) + b_1S(t), \\ \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} = -e_2u_2(t) + b_2I(t), \end{cases} \tag{1}$$

where $S(t)$ and $I(t)$ denote the susceptible and infected individuals, respectively; $u_1(t)$ and $u_2(t)$ are feedback control variables; all the coefficients are positive constants, r is the recruitment rate of susceptible population, μ is the death rate of the infected population, b is the transmission rate when susceptible individuals contact with infectious, and f denotes the intraspecific competition; b_i, c_i and e_i $(i = 1, 2)$ are control parameters. By introducing such indirect control variables, one may successfully alter the positions of positive equilibrium and retain its stability, see [2, 3].

Most of the works on epidemic models mainly focus on the homogeneous population, that is, all individuals are homogeneously mixed. In fact, spatial effects cannot be neglected in studying the spread of epidemics due to the large mobility of people [4, 5]. However, there have been few results on the combined influences of spatial diffusions and feedback controls. For this reason, we assume that these two kinds of people can move freely and consider the reaction-diffusion system of the form

$$\begin{cases} \frac{\partial S}{\partial t} = d_1 \frac{\partial^2 S}{\partial x^2} + S(r - a - bI - c_1 u_1), & x \in (0, l\pi), \quad t > 0, \\ \frac{\partial I}{\partial t} = d_2 \frac{\partial^2 I}{\partial x^2} + I(bS - \mu - fI - c_2 u_2), & x \in (0, l\pi), \quad t > 0, \\ \frac{\partial u_1}{\partial t} = d_3 \frac{\partial^2 u_1}{\partial x^2} - e_1 u_1 + b_1 S, & x \in (0, l\pi), \quad t > 0, \\ \frac{\partial u_2}{\partial t} = d_4 \frac{\partial^2 u_2}{\partial x^2} - e_2 u_2 + b_2 I, & x \in (0, l\pi), \quad t > 0, \\ \frac{\partial S(x,t)}{\partial x} = \frac{\partial I(x,t)}{\partial x} = \frac{\partial u_1(x,t)}{\partial x} = \frac{\partial u_2(x,t)}{\partial x} = 0, & x = 0, l\pi, \quad t > 0, \\ S(x,0) = \varphi_1(x) \geq 0, \ I(x,0) = \varphi_2(x) \geq 0, & x \in [0, l\pi], \\ u_1(x,0) = \varphi_3(x) \geq 0, \ u_2(x,0) = \varphi_4(x) \geq 0, & x \in [0, l\pi], \end{cases} \tag{2}$$

where d_i $(i = 1,2,3,4)$ are diffusion coefficients, and the initial functions $\varphi_i(x)$ $(i = 1,2,3,4)$ are Hölder continuous in $[0, l\pi] \times [0, +\infty)$. The homogeneous Neumann boundary condition means that the four variables have zero flux across the boundary.

It is of interest to study whether spatial diffusions change the dynamic properties of SI epidemic model with feedback controls. Motivated by these factors, the aim of this paper is to investigate the permanence and global stability of system (2). The methods used here can also be applicable to other feedback control models. The results here are novel, and to our knowledge, there are no similar results published for this diffusive epidemic model with feedback controls in existing literature. Based on these results, we may also give some useful suggestions to government or medical institution.

This paper is organized as follows. In Section 2, we study the well-posedness for system (2). In Section 3, we discuss the global attractor and permanence of solutions. In Section 4, we construct Lyapunov functions to investigate the global stability of two equilibria. Finally, some numerical examples and a brief discussion are given in Section 5.

2. **Preliminaries.**

In this section, we establish the existence, uniqueness, positivity and boundedness of solutions of (2) because this model describes the evolution of susceptible and infected population. Hence, the variables should remain nonnegative and bounded.

Theorem 2.1. *For system (2), there exists a unique solution defined on* $[0, +\infty)$ *and this solution remains nonnegative and uniformly bounded for all* $t \geq 0$.

Proof: By standard existence theory in [6, 7], it is easy to establish the local existence of the unique solution $(S(x,t), I(x,t), u_1(x,t), u_2(x,t))$ of system (2) for $x \in [0, l\pi]$ and $t \in [0, T_{\max})$, where $T_{\max}$ is the maximal existence time for solutions of (2).

We can also verify that $\mathbf{0} = (0,0,0,0)$ and $\mathbf{M_1} = (M_1, M_2, M_3, M_4)$ are a pair of coupled lower-upper solutions to problem (2), where $M_1 = \max\left\{\frac{r}{a}, \|\varphi_1\|_\infty\right\}$, $M_2 = \max\left\{\frac{bM_1}{f}, \|\varphi_2\|_\infty\right\}$, $M_3 = \max\left\{\frac{b_1 M_1}{e_1}, \|\varphi_3\|_\infty\right\}$, $M_4 = \max\left\{\frac{b_2 M_2}{e_2}, \|\varphi_4\|_\infty\right\}$.

According to the results in [6], we can easily derive that system (2) has exactly one solution $\mathbf{U}(x,t) = (S(x,t), I(x,t), u_1(x,t), u_2(x,t))$ in $[0, l\pi] \times [0, +\infty)$ satisfying $\mathbf{0} \leq \mathbf{U} \leq \mathbf{M}$. Then the solutions are uniformly bounded and we deduce that $t_{\max} = +\infty$. This proves the theorem.

As we know, spatial diffusion does not change the existence of constant equilibria. We restate the useful results from [1]. The basic reproduction number of the infection is given by $R_0 = \frac{(br - a\mu)e_1}{b_1 c_1 \mu}$. The following lemma presents the sufficient conditions for existence and uniqueness of an endemic equilibrium.

Lemma 2.1. *(1) If* $R_0 \leq 1$*, then system (2) has the unique disease-free equilibrium* $E^0 = (S^0, 0, u_1^0, 0)$*, where* $S^0 = \frac{e_1 r}{ae_1 + b_1 c_1}$, $u_1^0 = \frac{b_1 r}{ae_1 + b_1 c_1}$.

(2) If $R_0 > 1$, then system (2) has the endemic equilibrium $E^* = (S^*, I^*, u_1^*, u_2^*)$, where $S^* = \frac{be_2\mu + r(e_2 f + b_2 c_2)}{Ae_2}$, $I^* = \frac{be_1 r - ae_1\mu - b_1 c_1 \mu}{Ae_1}$, $A = b^2 + \left(a + \frac{b_1 c_1}{e_1}\right)\left(f + \frac{b_2 c_2}{e_2}\right)$, $u_1^* = \frac{b_1 S^*}{e_1}$ and $u_2^* = \frac{b_2 I^*}{e_2}$.

3. **Global Attractor and Permanence.** We first show that $\mathcal{R} = \left[0, \frac{r}{a}\right] \times \left[0, \frac{br}{af}\right] \times \left[0, \frac{d_1 r}{ae_1}\right] \times \left[0, \frac{bd_2 r}{ae_2 f}\right]$ is a global attractor for all solutions of system (2); in other words, any nonnegative solution $(S(x,t), I(x,t), u_1(x,t), u_2(x,t))$ of (2) lies in $\mathcal{R}$ as $t \to +\infty$ for all $x \in (0, l\pi)$.

Theorem 3.1. *(Dissipativeness) Let (S, I, u_1, u_2) be the unique solution of system (2). Then, for any $x \in [0, l\pi]$, we have*

$$\limsup_{t\to+\infty} S(x,t) \le \frac{r}{a}, \quad \limsup_{t\to+\infty} I(x,t) \le \frac{br}{af},$$

$$\limsup_{t\to+\infty} u_1(x,t) \le \frac{d_1 r}{ae_1}, \quad \limsup_{t\to+\infty} u_2(x,t) \le \frac{bd_2 r}{ae_2 f}.$$

Proof: From the first equation of system (2) and the positivity of solutions, we can obtain

$$\frac{\partial S}{\partial t} - d_1 \frac{\partial^2 S}{\partial x^2} \le S(r - aS) \qquad \text{for} \quad (x,t) \in (0, l\pi) \times [0, +\infty).$$

The comparison principle of parabolic equations in [7] shows that

$$\limsup_{t\to+\infty} S(x,t) \le \frac{r}{a}.$$

Then for an arbitrary $\varepsilon_1 > 0$, there exists $T_1 \in (0, +\infty)$ such that $S(x,t) \le \frac{r}{a} + \varepsilon_1$ for $(x,t) \in (0, l\pi) \times [T_1, +\infty)$. Thus, by the second equation of system (2), we get

$$\frac{\partial I}{\partial t} - d_2 \frac{\partial^2 I}{\partial x^2} \le I\left[b\left(\frac{r}{a} + \varepsilon_1\right) - fI\right].$$

This implies $\limsup_{t\to+\infty} I(x,t) \le \frac{br}{af}$ by the arbitrary of ε_1 and the standard comparison principle of parabolic equations. Similarly, we can also have

$$\limsup_{t\to+\infty} u_1(x,t) \le \frac{d_1 r}{ae_1}, \quad \limsup_{t\to+\infty} u_2(x,t) \le \frac{bd_2 r}{ae_2 f}.$$

The proof is complete.

Definition 3.1. *System (2) is said to be not persistent if*

$$\min\left\{\liminf_{t\to+\infty} S(x,t),\ \liminf_{t\to+\infty} I(x,t),\ \liminf_{t\to+\infty} u_1(x,t),\ \liminf_{t\to+\infty} u_2(x,t)\right\} = 0$$

for some of its nonnegative solutions. Otherwise, system (2) is said to be persistent.

Definition 3.2. *System (2) is said to be permanent if*

$$m \le \liminf_{t\to+\infty} S(x,t) \le \limsup_{t\to+\infty} S(x,t) \le M,\ m \le \liminf_{t\to+\infty} I(x,t) \le \limsup_{t\to+\infty} I(x,t) \le M,$$

$$m \le \liminf_{t\to+\infty} u_1(x,t) \le \limsup_{t\to+\infty} u_1(x,t) \le M,\ m \le \liminf_{t\to+\infty} u_2(x,t) \le \limsup_{t\to+\infty} u_2(x,t) \le M$$

for some of its nonnegative solutions, where m and M are positive constants.

In the next, we shall discuss the non-persistence and permanence of system (2).

Theorem 3.2. *If $br \le a\mu$, then system (2) is not persistent.*

Proof: From the proof of Theorem 3.1, we have more precise estimate as follows:

$$\frac{\partial I}{\partial t} - d_2 \frac{\partial^2 I}{\partial x^2} \leq I \left[b \left(\frac{r}{a} + \varepsilon_1 \right) - \mu - fI \right],$$

which yields that

$$\limsup_{t \to +\infty} I(x,t) \leq \frac{br - a\mu}{af} \leq 0.$$

Thus system (2) is not persistent due to Definition 3.2. This completes the proof.

Theorem 3.3. *Assuming that* $be_2 r \left(ae_1 f - b^2 e_1 - c_1 d_1 f \right) > ae_1 \left(ae_2 f \mu + bc_2 d_2 r \right)$, *then system (2) is permanent.*

Proof: From Theorem 3.1, for an arbitrary $\varepsilon_2 > 0$, there exists a common $T_2 > 0$ such that

$$S(x,t) \leq \frac{r}{a} + \varepsilon_2, \quad I(x,t) \leq \frac{br}{af} + \varepsilon_2, \quad u_1(x,t) \leq \frac{d_1 r}{ae_1} + \varepsilon_2, \quad u_2(x,t) \leq \frac{bd_2 r}{ae_2 f} + \varepsilon_2$$

in $(0, l\pi) \times [T_2, +\infty)$. Then we have

$$\frac{\partial S}{\partial t} - d_1 \frac{\partial^2 S}{\partial x^2} \geq S \left[r - \left(\frac{b^2 r}{af} + \varepsilon_2 \right) - c_1 \left(\frac{d_1 r}{ae_1} + \varepsilon_2 \right) - aS \right],$$

and

$$\liminf_{t \to +\infty} S(x,t) \geq \frac{r - \left(\frac{b^2 r}{af} + \varepsilon_2 \right) - c_1 \left(\frac{d_1 r}{ae_1} + \varepsilon_2 \right)}{a}.$$

Under the assumption and arbitrariness of ε_2, we can obtain

$$\liminf_{t \to +\infty} S(x,t) \geq \frac{r \left(ae_1 f - b^2 e_1 - c_1 d_1 f \right)}{a^2 e_1 f} > 0.$$

For appropriately small $\varepsilon_3 > 0$, there exists $T_3 > T_2$ such that

$$S(x,t) \geq \frac{r \left(ae_1 f - b^2 e_1 - c_1 d_1 f \right)}{a^2 e_1 f} - \varepsilon_3 > 0 \qquad \text{for} \quad (x,t) \in (0, l\pi) \times [T_3, +\infty).$$

Furthermore, we have

$$\frac{\partial I}{\partial t} - d_2 \frac{\partial^2 I}{\partial x^2} \geq I \left[\frac{br \left(ae_1 f - b^2 e_1 - c_1 d_1 f \right)}{a^2 e_1 f} - b\varepsilon_3 - \mu - c_2 \left(\frac{bd_2 r}{ae_2 f} + \varepsilon_2 \right) - fI \right],$$

and

$$\liminf_{t \to +\infty} I(x,t) \geq \frac{\frac{br\left(ae_1 f - b^2 e_1 - c_1 d_1 f\right)}{a^2 e_1 f} - b\varepsilon_3 - \mu - c_2 \left(\frac{bd_2 r}{ae_2 f} + \varepsilon_2 \right)}{f}.$$

By the continuity as $\varepsilon_2 \to 0$ and $\varepsilon_3 \to 0$, we obtain

$$\liminf_{t \to +\infty} I(x,t) \geq \frac{br \left(ae_1 f - b^2 e_1 - c_1 d_1 f \right)}{a^2 e_1 f^2} - \frac{\mu}{f} - \frac{bc_2 d_2 r}{ae_2 f^2}$$

According to similar procedure, we can also have

$$\liminf_{t \to +\infty} u_1(x,t) \geq \frac{b_1 r \left(ae_1 f - b^2 e_1 - c_1 d_1 f \right)}{a^2 e_1^2 f} > 0,$$

$$\liminf_{t \to +\infty} u_2(x,t) \geq \frac{bb_2 r \left(ae_1 f - b^2 e_1 - c_1 d_1 f \right)}{a^2 e_1 e_2 f^2} - \frac{b_2 \mu}{e_2 f} - \frac{bb_2 c_2 d_2 r}{ae_2^2 f^2} > 0.$$

Combining dissipativeness and definition of permanence, we can conclude that system (2) is permanent under the assumption. The proof is complete.

4. **Global Stability.** In this section, we discuss the global stability of nonnegative equilibria by constructing suitable Lyapunov functions.

Theorem 4.1. *If $R_0 < 1$, then the disease-free equilibrium E^0 is globally asymptotically stable.*

Proof: Assume that $(S(x,t), I(x,t), u_1(x,t), u_2(x,t))$ is a positive solution of system (2) and consider the following Lyapunov function:

$$W(t) = \int_\Omega \left[\left(S - S^0 - S^0 \ln \frac{S}{S^0}\right) + I + \frac{c_1}{2b_1}\left(u_1 - u_1^0\right)^2 + \frac{c_2}{2b_2}u_2^2\right] \mathrm{d}x.$$

Calculating the derivative along the solutions of (2), we have

$$\begin{aligned}
\frac{\mathrm{d}W(t)}{\mathrm{d}x} &= \int_\Omega \left[\frac{\partial S}{\partial t} - \frac{S}{S^0}\frac{\partial S}{\partial t} + \frac{\partial I}{\partial t} + \frac{c_1}{b_1}\left(u_1 - u_1^0\right)\frac{\partial u_1}{\partial t} + \frac{c_2}{b_2}u_2\frac{\partial u_2}{\partial t}\right] \mathrm{d}x \\
&= \int_\Omega \left[d_1\frac{S - S^0}{S}\Delta S + d_2 \Delta I + \frac{c_1 d_3}{2b_1}\left(u_1 - u_1^0\right)\Delta u_1 + \frac{c_2 d_4}{2b_2}u_2 \Delta u_2\right] \mathrm{d}x \\
&\quad + \int_\Omega \Big[\left(S - S^0\right)\left(-a\left(S - S^0\right) - bI - c_1\left(u_1 - u_1^0\right)\right) + \frac{c_2}{b_2}\left(-eu_2 + b_2 I\right) \\
&\quad + I\left(bS - \mu - fI - c_2 u_2\right) + \frac{c_1}{b_1}\left(u_1 - u_1^0\right)\left(-e_1\left(u_1 - u_1^0\right) + b_1\left(S - S^0\right)\right)\Big] \mathrm{d}x \\
&\triangleq I_1 + I_2.
\end{aligned}$$

According to the Green's formula and zero-flux boundary condition, we can obtain

$$I_1 = -\int_\Omega \frac{S^0 |\nabla S|^2}{S^2} \mathrm{d}x - \int_\Omega \frac{c_1 d_3 u_1}{2b_1}\frac{u_1^0 |\nabla u_1|^2}{u_1^2} \mathrm{d}x \leq 0.$$

On the other side, we have

$$I_2 = \int_\Omega \left[-a\left(S - S^0\right)^2 - I\left(\mu - \frac{bre_1}{ae_1 + b_1 c_1}\right) - \frac{c_1 e_1}{b_1}\left(u_1 - u_1^0\right)^2 - \frac{c_2 e_2}{b_2}u_2^2\right] \mathrm{d}x.$$

Then we have from the assumption $R_0 < 1$ that $I_2 \leq 0$. It is obvious that $W'(t) \leq 0$ and the equality holds if and only if $S = S^0$, $I = 0$, $u_1 = u_1^0$ and $u_2 = 0$. Thus, the proof is completed by LaSalle's invariance principle.

Theorem 4.2. *If $R_0 > 1$, then the endemic equilibrium E^* is globally asymptotically stable.*

Proof: We construct the following Lyapunov function

$$L(x,t) = \int_\Omega \left[S - S^* - S^* \ln \frac{S}{S^*} + \frac{c_1}{2b_1}\left(u_1 - u_1^*\right)^2 + I - I^* - I^* \ln \frac{I}{I^*} + \frac{c_2}{2b_2}\left(u_2 - u_2^*\right)^2\right] \mathrm{d}x.$$

Using the similar techniques in the proof of Theorem 4.1, we get

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= -\int_\Omega \left(\frac{d_1 S^* |\nabla S|^2}{S^2} + \frac{d_1 I^* |\nabla I|^2}{I^2} + \frac{c_1 d_3 u_1^*}{b_1 u_1}|\nabla u_1|^2 + \frac{c_2 d_4 u_2^*}{b_2 u_2}|\nabla u_2|^2\right) \mathrm{d}x \\
&\quad - \int_\Omega \left[a\left(S - S^*\right)^2 + f\left(I - I^*\right)^2 + \frac{c_1 e_1}{b_1}\left(u_1 - u_1^*\right)^2 + \frac{c_2 e_2}{b_2}\left(u_2 - u_2^*\right)^2\right] \mathrm{d}x.
\end{aligned}$$

Hence, $\frac{\mathrm{d}L}{\mathrm{d}t} \leq 0$. It follows from LaSalle's invariance principle that the positive equilibrium E^* is globally asymptotically stable when $R_0 > 1$.

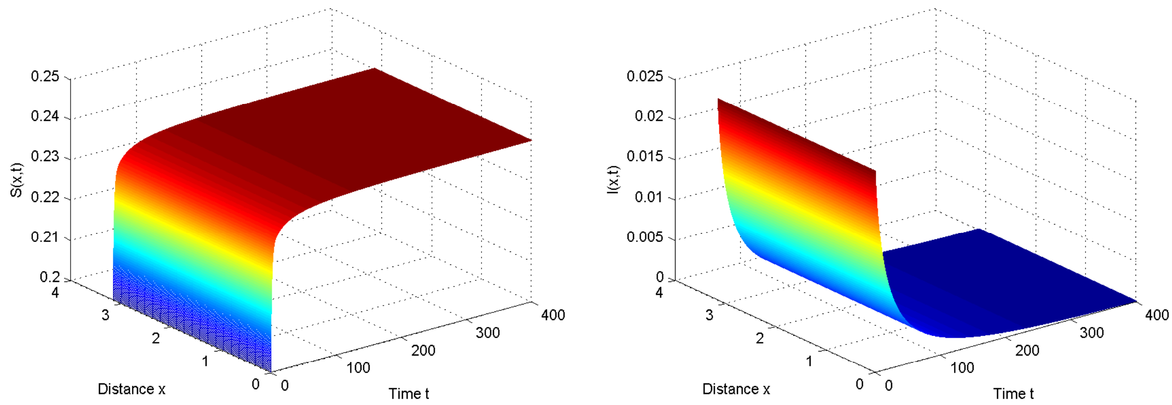

FIGURE 1. The disease-free equilibrium E^0 is globally stable when $R_0 < 1$.

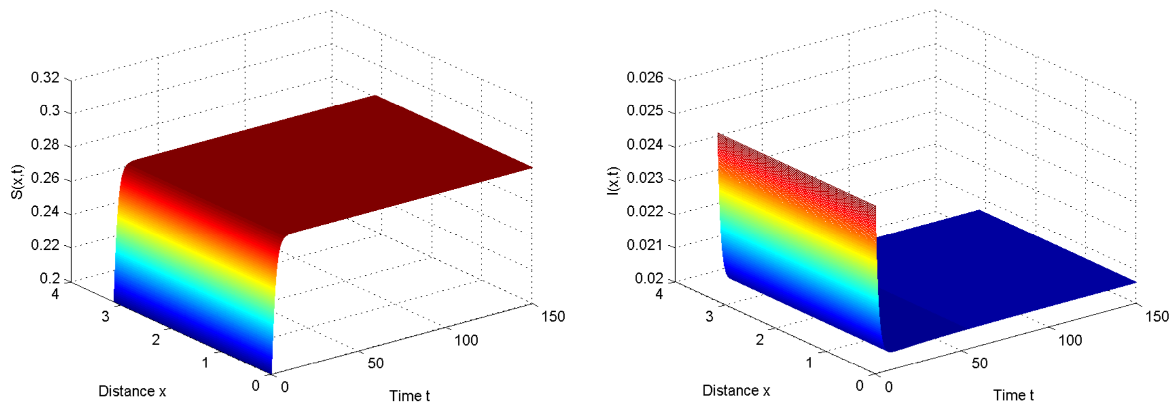

FIGURE 2. The endemic equilibrium E^* is globally stable when $R_0 > 1$.

5. **Numerical Examples and Conclusions.** In this section, we give some numerical simulations for system (2). According to the positivity of parameters in previous assumptions, we choose the coefficients as [1] and set $r = 0.6$, $a = e_2 = 2$, $b = f = b_2 = c_2 = 1$, $\mu = 0.25$ and $l = 1$. When $e_1 = 2$ and $b_1 = c_1 = 1$, we obtain $R_0 = 0.8 < 1$ and the disease-free equilibrium $E^0 = (0.24, 0, 0.12, 0)$ is globally asymptotically stable (see Figure 1). This means that the disease will eventually vanish when the parameters e_1, b_1 and c_1 are suitably selected. If $e_1 = 4$ and $b_1 = c_1 = 0.5$, then $R_0 = 6.4 > 1$ and the endemic equilibrium $E^* = (0.2809, 0.02061, 0.03511, 0.01031)$ is globally asymptotically stable (see Figure 2). Practically speaking, when the control decrease coefficient e_1 is larger and the feedback control coefficients b_1 and c_1 are slightly smaller, the disease will not be extinct. Moreover, the number of healthy and infected individuals will achieve positive constant steady states fleetly.

In this paper, we have proposed a diffusive SI epidemic model with feedback controls by introducing the spatial diffusion and zero-flux boundary condition. It is shown that the disease-free equilibrium and endemic equilibrium are globally asymptotically stable when $R_0 < 1$ or $R_0 > 1$ respectively. This means that there is no Turing pattern and self-diffusion has no influence on the global stability. More precisely, the healthy people would be to keep away from the infected ones [8, 9]. The movement of susceptible people is in the direction of lower concentration of the infected, which can be mathematically described by cross diffusion. To explore more complex dynamics of system (2), in the future, we shall take into account the cross-diffusion factor and discuss the pattern formation.

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