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FURTHER RESULT ON STABILITY ANALYSIS OF DISCRETE-TIME SYSTEMS WITH INTERVAL TIME-VARYING DELAY AND LINEAR FRACTIONAL PERTURBATIONS VIA DELAY-PARTITIONING APPROACH

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Abstract. *The main goal of this paper is to investigate the problem of asymptotic stability of uncertain discrete-time systems involving interval time-varying delay. The uncertainties of the system matrices are assumed to be structured fractional uncertainty. By the construction of suitable Lyapunov-Krasovskii functionals with some double summation terms and the utilization of discrete Jensen inequality, a delay-dependent sufficient condition for calculating the maximum allowable upper bound of the delay interval is derived to achieve asymptotic stability of such systems. The stability criterion is expressed in terms of linear matrix inequality (LMI) which can be easily checked by using the standard numerical tools for MATLAB. In addition, a delay-partitioning approach combining Park and Jensen inequalities will be used to improve the stability condition of proposed results. Finally, a numerical example is provided to show the usefulness and reduce conservativeness of the proposed result.*
Keywords: Discrete-time systems, Asymptotic stability, Interval time-varying delay, Discrete Jensen inequality, Delay-partitioning approach, Park inequality

1. **Introduction.** Time-delay phenomenon is often confronted in various practical systems, such as aircraft stabilization, chemical engineering systems, congestion control in high speed networks, hydraulic systems, inferred grinding model, manual control, neural network, nuclear reactor, population dynamic model, rolling mill, ship stabilization, and systems with lossless transmission lines. Existence of delay in a system may cause a source of instability or bad performance in closed loop control systems. Hence, stability problem for continuous-time and discrete-time systems with time delay has received some attention by many researchers in recent years [1]. In this paper, we will focus on stability analysis of discrete-time systems with interval time-varying delay [2-11].

The models of system always contain some uncertain elements because exact mathematical models are difficult to construct; such as additive environmental noises, ageing of systems, poor plant knowledge. Many types of perturbations had been considered in the past. Linear fractional perturbations in [12,13] are the generalized perturbed forms versus parameter perturbations in [9]. Hence, the robust asymptotic stability problem for discrete-time systems with linear fractional perturbations is considered in this paper. In recent years, many approaches had been used to improve the stability for discrete time-delay systems. Delay-partitioning approach is a new developed tool to decompose the bound of the interval time-varying delay into p uniform subintervals [8,9]. In those previous results, the dimension of LMI conditions will be large due to the large number of partition in interval time-varying delay. In this paper, the novel delay-partitioning approach is proposed with the same dimension in LMI conditions. Discrete Jensen and Park inequalities are used to derive our main results in this paper. More accurate evaluation is the main developed tool in each subinterval of interval time-varying delay [14,15]. In this paper, a new Lyapunov functional is proposed without including time-varying delay $r(k)$. Hence, some upper bounds evaluations in [13] can be avoided. In this paper, a delay-dependent LMI condition is proposed to guarantee the robust asymptotic stability. A numerical example is provided to show the effectiveness of the proposed results.

The remainder of this paper is organized as follows. The problem formulation and the main results are given in Section 2. Section 3 provides a numerical example to illustrate the main results. Finally, a conclusion is made in Section 4.

Notations. For a matrix A, we denote the transpose by A^T, symmetric positive (negative) definite by $A > 0$ ($A < 0$). $A \leq B$ means that matrix $B - A$ is symmetric positive semi-definite. 0 and I denote the zero matrix and identity matrix, respectively. $diag[\cdots]$ stands for a block-diagonal matrix.

2. Problem Formulation and Main Results.

Consider the following uncertain discrete-time system with interval time-varying delay and linear fractional perturbations

$$x(k+1) = (A + \Delta A(k))x(k) + (B + \Delta B(k))x(k - r(k)), \tag{1a}$$

$$x(k) = \phi(k), \quad k \in [-r_M, 0], \tag{1b}$$

where $x(k) \in \Re^n$ is the state vector, A and $B \in \Re^{n\times n}$ are some given constant matrices, and $\phi(k) \in \Re^n$ is an initial state function. The time-varying delay $r(k)$ is a function from $\{0, 1, 2, 3, \ldots\}$ to $\{1, 2, 3, \ldots\}$, satisfying the following condition:

$$0 < r_m \leq r(k) \leq r_M, \tag{2}$$

where r_m and r_M are two given positive integers, respectively. $\Delta A(k)$ and $\Delta B(k)$ are two perturbed matrices and satisfy the following conditions

$$[\Delta A(k) \quad \Delta B(k)] = M \cdot \Delta(k) \cdot [N_1 \quad N_2], \tag{3a}$$

$$\Delta(k) = [I - \Gamma(k)\Xi]^{-1}\Gamma(k), \quad \Xi\Xi^T < I, \tag{3b}$$

where M, N_1, N_2, and Ξ are some given constant matrices of compatible dimensions, and $\Gamma(k)$ is unknown matrix representing the parameter perturbations which satisfy

$$\Gamma^T(k)\Gamma(k) \leq I. \tag{3c}$$

Remark 2.1. *In the system (1), these uncertainties in (3a)-(3c) are referred to as a linear fractional perturbations [12]. This class of uncertainties has been investigated in many classes of systems; such as fuzzy systems; neural networks; switched system [12,13,16]. It is easy to see that when* $\Xi = 0$*, linear fractional perturbations in (3a)-(3c) are reduced to norm bounded ones. Hence, we assume that the uncertainties of system under consideration satisfy the linear fractional perturbation conditions in (3a)-(3c) with some given constant matrices* M*,* N_1*,* N_2*, and* Ξ*.*

The following lemmas are introduced to derive our main results.

Lemma 2.1. [14] *(Discrete Jensen inequality) For any matrix* $R \in \Re^{n\times n} > 0$*, positive integers* $r_2 < r_1$*, vector function* $\omega(\theta) \in \Re^n$*, the following inequalities are satisfied:*

$$-(r_2 - r_1) \cdot \sum_{\theta=k-r_2}^{k-r_1-1} \omega^T(\theta)R\omega(\theta) \leq -\left[\sum_{\theta=k-r_2}^{k-r_1-1} \omega(\theta)\right]^T R \left[\sum_{\theta=k-r_2}^{k-r_1-1} \omega(\theta)\right],$$

and

$$-\frac{(r_2-r_1)\cdot(r_2-r_1+1)}{2}\cdot\sum_{l=-r_2}^{-r_1-1}\sum_{\theta=k+l}^{k-r_1-1}\omega^T(\theta)R\omega(\theta)$$
$$\leq-\left[\sum_{l=-r_2}^{-r_1-1}\sum_{\theta=k+l}^{k-r_1-1}\omega(\theta)\right]^T R\left[\sum_{l=-r_2}^{-r_2-1}\sum_{\theta=k+l}^{k-r_1-1}\omega(\theta)\right].$$

Lemma 2.2. [15] *(Park inequality) For any matrices* $V\in\Re^{n\times n}>0$, $M_1,M_2\in\Re^{n\times m}$, *a positive real number* $0<\alpha<1$, *vector* $\omega\in\Re^m$, *there exists a matrix* $X\in\Re^{n\times n}$, *such that*

$$\begin{bmatrix} V & X \\ * & V \end{bmatrix}>0.$$

Then the following inequality is satisfied:

$$-\left[\frac{1}{\alpha}\omega^T M_1^T V M_1\omega+\frac{1}{1-\alpha}\omega^T M_2^T V M_2\omega\right]\leq-\omega^T\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}^T\begin{bmatrix} V & X \\ * & V \end{bmatrix}\begin{bmatrix} M_1 \\ M_2 \end{bmatrix}\omega.$$

Lemma 2.3. [12] *For the matrix* $\Delta(k)$ *defined in (3b) and (3c), the following statements are equivalent for any real matrices* U, W *and* X *with* $X=X^T$*:*

(I) The inequality is satisfied

$$X+U\Delta(k)W+W^T\Delta^T(k)U^T<0,$$

(II) There exists a scalar $\varepsilon>0$, *such that*

$$\begin{bmatrix} X & U & \varepsilon\cdot W^T \\ * & -\varepsilon\cdot I & \varepsilon\cdot\Xi^T \\ * & * & -\varepsilon\cdot I \end{bmatrix}<0,$$

where the matrix Ξ *is defined in (3b).*

Now, when the whole delay interval is decomposed into p delay subintervals, i.e., $0<r_m=r_0<r_1<r_2<\cdots<r_{p-1}<r_p=r_M$, where r_i, $i=0,1,2,\ldots,p$, are some positive integers, a novel delay-dependent stability criterion will be proposed to guarantee the asymptotic stability of uncertain discrete-time system with interval delay.

Theorem 2.1. *For some selected positive integers* $0<r_m=r_0<r_1<r_2<\cdots<r_{p-1}<r_p=r_M$, *the system (1) with (2) and (3) is asymptotically stable, if there exist* $n\times n$ *matrices* $P>0$, $R_i>0$, $i=1,2,3,4$, $S_j>0$, $H_j>0$, $L_j>0$, $j=1,2$, $W_q>0$, $q=1,2,3$, *and any* $n\times n$ *matrices* T, X, Y, Z, *satisfying*

$$R_1+W_1>0,\quad R_2+W_2>0,\quad R_2+W_3>0,\quad\begin{bmatrix} H_1 & W_1 \\ * & R_1+W_1+L_1 \end{bmatrix}>0, \tag{4a}$$

$$\begin{bmatrix} H_2 & W_2 \\ * & L_2 \end{bmatrix}>0,\quad\begin{bmatrix} H_2 & W_3 \\ * & L_2 \end{bmatrix}>0,\quad\begin{bmatrix} H_2 & W_2 & T & Y \\ * & L_2 & X & Z \\ * & * & H_2 & W_3 \\ * & * & * & L_2 \end{bmatrix}>0, \tag{4b}$$

and a scalar $\varepsilon>0$, *such that the following LMI conditions are feasible:*

$$\Sigma_i=\begin{bmatrix} \Sigma_{1i} & \Sigma_2 \\ * & \Sigma_3 \end{bmatrix}<0,\quad i=1,2,\ldots,p, \tag{4c}$$

where

$$\Sigma_{1i} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & 0 & 0 & \Sigma_{15} & 0 & 0 \\ * & \Sigma_{22i} & \Sigma_{23i} & \Sigma_{24} & \Sigma_{25} & \Sigma_{26} & \Sigma_{27} \\ * & * & \Sigma_{33i} & \Sigma_{34i} & 0 & \Sigma_{36} & \Sigma_{37} \\ * & * & * & \Sigma_{44i} & 0 & \Sigma_{46} & \Sigma_{47} \\ * & * & * & * & \Sigma_{55} & 0 & 0 \\ * & * & * & * & * & \Sigma_{66} & \Sigma_{67} \\ * & * & * & * & * & * & \Sigma_{77} \end{bmatrix},$$

$$\Sigma_2 = \begin{bmatrix} \Sigma_{18} & \Sigma_{19} & 0 & \Sigma_{111} \\ 0 & 0 & 0 & 0 \\ \Sigma_{38} & \Sigma_{39} & 0 & \Sigma_{311} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Sigma_3 = \begin{bmatrix} \Sigma_{88} & 0 & \Sigma_{810} & 0 \\ * & \Sigma_{99} & \Sigma_{910} & 0 \\ * & * & \Sigma_{1010} & \Sigma_{1011} \\ * & * & * & \Sigma_{1111} \end{bmatrix}, \tag{4d}$$

with

$\Sigma_{11} = -P + S_1 + r_m^2 \cdot H_1 + r_{Mm}^2 \cdot H_2 + (r_m - 1) \cdot W_1 - R_1 - r_m^2 \cdot R_3 - L_1$,
$\Sigma_{12} = R_1 + W_1 + L_1$, $\Sigma_{15} = r_m \cdot R_3 - W_1^T$, $\Sigma_{18} = (A - I)^T \Theta$,
$\Sigma_{19} = A^T P$, $\Sigma_{111} = \varepsilon \cdot N_1^T$,
$\Sigma_{22i} = -S_1 + S_2 - (r_m + 1) \cdot W_1 - R_1 + r_{Mm} \cdot W_2 - \lambda_i \cdot (R_2 + W_2) - r_{Mm}^2 \cdot R_4 - L_1 - L_2$,
$\Sigma_{23i} = \lambda_i \cdot (R_2 + W_2) + L_2 - Z$, $\Sigma_{24} = Z$, $\Sigma_{25} = W_1^T$,
$\Sigma_{26} = r_{Mm} \cdot R_4 - W_2^T$, $\Sigma_{27} = r_{Mm} \cdot R_4 - X$,
$\Sigma_{33i} = r_{Mm} \cdot (-W_2 + W_3) - \lambda_i \cdot (R_2 + W_2) - \gamma_i \cdot (R_2 + W_3) + Z + Z^T - 2L_2$,
$\Sigma_{34i} = \gamma_i \cdot (R_2 + W_3) - Z + L_2$, $\Sigma_{36} = W_2^T - Y^T$, $\Sigma_{37} = X - W_3^T$,
$\Sigma_{38} = B^T \Theta$, $\Sigma_{39} = B^T P$, $\Sigma_{311} = \varepsilon \cdot N_2^T$,
$\Sigma_{44i} = -(S_2 + r_{Mm} \cdot W_3) - \gamma_i \cdot (R_2 + W_3) - L_2$, $\Sigma_{46} = Y^T$, $\Sigma_{47} = W_3^T$,
$\Sigma_{55} = -R_3 - H_1$, $\Sigma_{66} = -R_4 - H_2$, $\Sigma_{67} = -R_4 - T$, $\Sigma_{77} = -R_4 - H_2$,
$\Sigma_{88} = -\Theta$, $\Sigma_{810} = \Theta^T M$, $\Sigma_{99} = -P$, $\Sigma_{910} = PM$, $\Sigma_{1010} = -\varepsilon \cdot I$,
$\Sigma_{1011} = \varepsilon \cdot \Xi^T$, $\Sigma_{1111} = -\varepsilon \cdot I$, $r_{Mm} = r_M - r_m$, $\hat{r} = \dfrac{r_m}{2} \cdot (r_m + 1)$,
$\tilde{r} = \dfrac{r_{Mm}}{2} \cdot (r_{Mm} + 1)$, $\lambda_i = \dfrac{r_{Mm}}{r_i - r_m}$, $\gamma_i = \dfrac{r_{Mm}}{r_M - r_{i-1}}$, $i = 1, 2, \ldots, p$,
$\Theta = r_m^2 \cdot (R_1 + L_1) + r_{Mm}^2 \cdot (R_2 + L_2) + \hat{r}^2 \cdot R_3 + \tilde{r}^2 \cdot R_4$.

Proof: Define the following Lyapunov functional:

$$V(x_k) = V_1(x_k) + V_2(x_k) + V_3(x_k), \tag{5a}$$

where

$$V_1(x_k) = x^T(k) P x(k) + \sum_{\theta=k-r_m}^{k-1} x^T(\theta) S_1 x(\theta) + \sum_{\theta=k-r_M}^{k-r_m-1} x^T(\theta) S_2 x(\theta), \tag{5b}$$

$$\begin{aligned} V_2(x_k) = {} & r_m \cdot \sum_{l=-r_m}^{-1} \sum_{\theta=k+l}^{k-1} \eta^T(\theta) R_1 \eta(\theta) + r_{Mm} \cdot \sum_{l=-r_M}^{-r_m-1} \sum_{\theta=k+l}^{k-1} \eta^T(\theta) R_2 \eta(\theta) \\ & + r_m \cdot \sum_{l=-r_m}^{-1} \sum_{\theta=k+l}^{k-1} y^T(\theta) \hat{R}_1 y(\theta) + r_{Mm} \cdot \sum_{l=-r_M}^{-r_m-1} \sum_{\theta=k+l}^{k-1} y^T(\theta) \hat{R}_2 y(\theta), \end{aligned} \tag{5c}$$

$$V_3(x_k) = \hat{r} \cdot \sum_{l=-r_m}^{-1} \sum_{\theta=k+l}^{k-1} (\theta - k - l + 1) \cdot \eta^T(\theta) R_3 \eta(\theta)$$

$$+\tilde{r}\cdot\sum_{l=-r_M}^{-r_m-1}\sum_{\theta=k+l}^{k-r_m-1}(\theta-k-l+1)\cdot\eta^T(\theta)R_4\eta(\theta)$$
$$+\tilde{r}^2\cdot\sum_{\theta=k-r_m}^{k-1}\eta^T(\theta)R_4\eta(\theta), \tag{5d}$$

where $P>0$, $R_i>0$, $i=1,2,3,4$, $S_j>0$, $H_j>0$, $L_j>0$, $\hat{R}_j=diag[H_j,L_j]>0$, $j=1,2$, $\eta(\theta)=x(\theta+1)-x(\theta)$, and $y(\theta)=\begin{bmatrix}x(\theta)^T & \eta(\theta)^T\end{bmatrix}^T$. Due to the page limit, this proof can be finished by [15,16].

3. Example.

Example 3.1. *Consider the system (1) with (2), no perturbation, and the following parameters [4]:*

$$A=\begin{bmatrix}0.8 & 0\\ 0.05 & 0.9\end{bmatrix},\quad B=\begin{bmatrix}-0.1 & 0\\ -0.2 & -0.1\end{bmatrix}. \tag{6}$$

By using the LMI Toolbox of Matlab, some allowable delay upper bounds that guarantee the asymptotic stability for system (1) with (2) and (6) are provided in Table 1. From Table 1, we can see that the presented stability results in this paper are less conservative than ones in [3,4,6-8,11,13].

Table 1. The allowable delay upper bound r_M for some r_m (without perturbations)

Results	$r_m=4$	$r_m=6$	$r_m=10$	$r_m=15$
[4]	8	9	12	16
[3]	13	14	15	18
[11]	13	14	17	20
[6]	15	16	18	21
[7]	15	16	18	21
[13]	17	17	17	19
[8] (p = 2)	17	*	*	*
[8] (p = 4)	18	*	*	*
Theorem 2.1 (p = 6)	20	20	22	25

* This sign represents that the original article does not provide the delay upper bound r_M.

From this numerical example, some other observations can be concluded from simulation:

(1) Better results may be achieved by more partitions on delay interval.
(2) Better results in Theorem 2.1 can be obtained by combining the Park and Jensen inequalities.
(3) Less LMI variables than nonnegative inequality approach in [13] can be obtained in the proposed approach.

4. **Conclusion.** In this paper, the robust asymptotic stability for uncertain discrete-time systems with interval time-varying delay and linear fractional perturbations has been considered. Some novel stability criteria have been developed via delay-partitioning approach. The obtained results have been shown to be less conservative than some existing published results in a numerical example. It is worth pointing out that the derived results can be extended to some more general discrete complex dynamical delayed neural networks. This will be discussed in the near future.

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REFERENCES

[1] K. Gu, V. L. Kharitonov and J. Chen, *Stability of Time-Delay Systems*, Birkhauser, Boston, Massachusetts, 2003.

[2] E. Fridman and U. Shaked, Stability and guaranteed cost control of uncertain discrete delay systems, *International Journal of Control*, vol.78, pp.235-246, 2005.

[3] H. Gao and T. Chen, New results on stability of discrete-time systems with time-varying state delay, *IEEE Trans. Automatic Control*, vol.52, pp.328-334, 2007.

[4] H. Gao, J. Lam, C. Wang and Y. Wang, Delay-dependent output feedback stabilisation of discrete-time systems with time varying state delay, *IEE Proc. of Control Theory and Applications*, vol.151, pp.691-698, 2004.

[5] L. Hetel, J. Daafouz and C. Iung, Equivalence between the Lyapunov-Krasovskii functions approach for discrete delay systems and that of stability conditions for switched systems, *Nonlinear Analysis: Hybrid Systems*, vol.2, pp.697-705, 2008.

[6] H. Huang and G. Feng, Improved approach to delay-dependent stability analysis of discrete-time systems with time-varying delay, *IET Control Theory and Applications*, vol.4, pp.2152-2159, 2010.

[7] P. Kokil, V. K. R. Kandanvli and H. Kar, Delay-partitioning approach to stability analysis of linear discrete-time systems with interval-like time-varying delay, *International Journal of Engineering Mathematics*, vol.2013, 2013.

[8] X. Meng, J. Lam, B. Du and H. Gao, A delay-partitioning approach to the stability analysis of discrete-time systems, *Automatica*, vol.46, pp.610-614, 2010.

[9] M. N. A. Parlakci, Robust stability of linear uncertain discrete-time systems with interval time-varying delay, *Turkish Journal of Electrical Engineering & Computer Sciences*, vol.22, pp.650-662, 2014.

[10] C. Peng and Y. C. Tian, Improved delay-dependent robust stability criteria for uncertain systems with interval time-varying delay, *IET Control Theory and Applications*, vol.2, pp.752-761, 2008.

[11] B. Zhang, S. Xu and Y. Zou, Improved stability criterion and its applications in delayed controller design for discrete-time systems, *Automatica*, vol.44, pp.2963-2967, 2008.

[12] T. Li, L. Guo and C. Sun, Robust stability for neural networks with time-varying delays and linear fractional uncertainties, *Neurocomputing*, vol.71, pp.421-427, 2007.

[13] C. H. Lien, K. W. Yu, J. D. Chen and L. Y. Chung, Sufficient conditions for global exponential stability of discrete switched time-delay systems with linear fractional perturbations via switching signal design, *Advances in Difference Equations*, Springer International Publishing AG, 2013.

[14] X. L. Zhu and G. H. Yang, Jensen inequality approach to stability analysis of discrete time systems with time-varying delay, *Proc. of American Control Conference*, pp.1644-1649, 2008.

[15] P. G. Park, J. W. Ko and C. Jeong, Reciprocally convex approach to stability of systems with time-varying delays, *Automatica*, vol.47, pp.235-238, 2011.

[16] C. H. Lien, K. W. Yu, J. H. Yang, I. C. Chou, J. D. Chen and L. Y. Chung, Novel delay-partitioning approach on stability of uncertain discrete switched time-delay systems via switching signal design, *IMA Journal of Mathematical Control and Information*, in press.