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CONSTRUCTION OF FRAMELET PACKETS ON $\mathbb{Z}$

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Abstract. *Wavelet frame packets or framelet packets based on wavelet frames on $L^2(\mathbb{R})$ have been well studied in theory and applications, since they can provide adaptive choice from a library of wavelet frames for a wide range of practically oriented tasks. However, the studies of wavelet frame packets in $l^2(\mathbb{Z})$ have been less involved. In this paper, we first present a scheme to construct a class of J-stage framelet packets, and then give an example on $\mathbb{Z}$ to explain our scheme.*

Keywords: Framelet packet, Convolution, First-stage framelet, Sequence space

1. **Introduction.** The dyadic wavelet frames have played an important role in the applications such as signal processing, communications, and sensing. For example, for a band-limited framelet ψ, the measure of supp $\left(\hat{\psi}_{j,k}\right)$ is 2^j times the measure of supp $\left(\hat{\psi}\right)$. Thus, the wavelet frames have poor frequency localization when j is large. For some applications, especially for speech signal processing, it is more convenient to have wavelet frames with better frequency localization. This will be provided by the framelet packets (or called wavelet frame packets).

The original idea of wavelet packets was introduced by Coifman et al. in [1,2]. However, the theory itself is worthy of further study. Some developments in the wavelet packets theory should be mentioned, such as multiwavelet packets [3] on $\mathbb{R}^d$, the non-tensor-product version [4] of wavelet packets on $\mathbb{R}^d$. Recently, using the so-called splitting trick given by Daubechies [5], Lu and Fan in [6,7] constructed a class of tight framelet packets with $2I_d$-dilation for $L^2(\mathbb{R}^d)$ from the unitary extension principles given by Ron and Shen in [8].

However, many algorithmic realizations based function systems given above in applied mathematics are in the digital setting because the input/output data and all filters are of discrete nature, and one of the most common shortcomings of some of such function systems is lack of providing a unified treatment of the continuum and digital settings, i.e., allowing a digital theory to be a natural digitization of the continuum theory. Curvelets, for instance, are known to yield tight frames but the digital curvelet transform is not designed within the curvelet-framework and hence, in particular, is not covered by the available theory [9]. In this paper, we will directly study the discrete versions of framelet packets on $\mathbb{Z}$ and their key properties.

This paper is organized as follows. Section 2 gives some notations and definitions we shall use. Section 3 gives a scheme to construct a class of J-stage framelet packets. We end this paper with an example on $\mathbb{Z}$ in Section 4.

2. **Preliminaries.** We begin by introducing some notation and definitions we shall use.

$\mathbb{H}$ denotes a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$ for each $x \in \mathbb{H}$. Let J be a numerable index set. A countable system $\{\phi_j\}_{j \in J}$ in $\mathbb{H}$ is called a frame for $\mathbb{H}$ if there exist constants A and B, $0 < A \leq B < \infty$, such that

$$A\|x\|^2 \leq \sum_{j \in J} |\langle x, \phi_j \rangle|^2 \leq B\|x\|^2 \tag{1}$$

holds for all $x \in \mathbb{H}$. The numbers A and B are called the lower and upper frame bounds, respectively. The frame bounds are not unique because numbers less than A and greater than B are also valid frame bounds. The optimal lower bound is the supremum of all lower bounds and the optimal upper bound is the infimum of all upper bounds. If $A = B = 1$, then the frame is called a Parseval frame.

We denote the inner product in $L^2([0, 2\pi])$ by $\langle f, g \rangle_{L^2([0,2\pi])} := \int_{[0,2\pi]} f(x)\overline{g(x)}dx/(2\pi)$, where $f, g \in L^2([0, 2\pi])$, its corresponding norm by $\| \cdot \|_{L^2([0,2\pi])} := \langle \cdot, \cdot \rangle^{1/2}_{L^2([0,2\pi])}$. For a sequence $u \in l^2(\mathbb{Z})$ we denote the j-th coordinate by $u(j)$, and call it a finitely supported sequence if only finitely many non-zero elements in it. Denote the inner product and the norm in $l^2(\mathbb{Z})$ by $\langle u, v \rangle = \sum_{k \in \mathbb{Z}} u(k)\overline{v(k)}$, where $u, v \in l^2(\mathbb{Z})$, and $\| \cdot \| := \langle \cdot, \cdot \rangle^{1/2}$, respectively. For a sequence $u \in l^2(\mathbb{Z})$, we define its Fourier transform by $\hat{u}(\xi) = \sum_{k \in \mathbb{Z}} u(k)e^{ik\xi}$ for almost all $\xi \in \mathbb{R}$. And the Plancherel formula says

$$\langle \hat{u}, \hat{v} \rangle_{L^2([0,2\pi])} = \langle u, v \rangle = \sum_{k \in \mathbb{Z}} u(k)\overline{v(k)}, \quad u, v \in l^2(\mathbb{Z}). \tag{2}$$

The discrete convolution $u * v$ of $u = \{u(k)\}_{k \in \mathbb{Z}}$ and $v = \{v(k)\}_{k \in \mathbb{Z}}$ is defined by

$$u * v(n) = \sum_{k \in \mathbb{Z}} u(n-k)v(k), \quad n \in \mathbb{Z}. \tag{3}$$

It is well known that $u * v \in l^2(\mathbb{Z})$ when $u \in l(\mathbb{Z})$ and $v \in l^2(\mathbb{Z})$. As usual, $l(\mathbb{Z}) := \big\{u = \{u(k)\}_{k \in \mathbb{Z}} : \sum_{k \in \mathbb{Z}} |u(k)| < +\infty\big\}$. It can be checked that $\widehat{u * v}(\xi) = \hat{u}(\xi)\hat{v}(\xi)$ for $u, v \in l^2(\mathbb{Z})$.

For a sequence $u = (u(n))_{n \in \mathbb{Z}}$, define the downsampling operator $D : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ and the upsampling operator $U : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ on $\mathbb{Z}$ by

$$D(u)(n) = u(2n), \quad U(u)(n) = \begin{cases} u(n/2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{4}$$

respectively. Clearly the m-fold composition of D with itself is denoted D^m, and similarly for U^m. Then

$$D^m(u)(n) = u(2^m n), \quad U^m(u)(n) = \begin{cases} u(n/2^m) & \text{if } n = 2^m j \text{ for some } j \in \mathbb{Z}, \\ 0 & \text{if } n \text{ is not divisible by } 2^m. \end{cases} \tag{5}$$

Definition 2.1. *For $k \in \mathbb{Z}$, the translation operator $R_k : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ is defined by $R_k u(n) = u(n-k)$ for all $n \in \mathbb{Z}$.*

Definition 2.2. *Suppose $u \in l^2(\mathbb{Z})$. For $n, k \in \mathbb{Z}$, define the conjugate reflection of u: $\tilde{u}(n) = \overline{u(-n)}$. Also define $u^*(n) = (-1)^n u(n)$.*

A simple calculation can lead to the following results.

Lemma 2.1. *Suppose $u, v \in l^2(\mathbb{Z})$. Then*
(a) $\tilde{u}, u^* \in l^2(\mathbb{Z})$, and $R_k u \in l^2(\mathbb{Z})$, $k \in \mathbb{Z}$.
(b) $(\tilde{u})^\wedge(\xi) = \overline{\hat{u}(\xi)}$; $(u^*)^\wedge(\xi) = \hat{u}(\xi + \pi)$; $\hat{\delta}(\xi) = 1$, where δ is the delta function.
(c) $(R_k u)^\wedge(\xi) = e^{ik\xi}\hat{u}(\xi)$; $\langle R_j u, R_k v \rangle = \langle u, R_{k-j} v \rangle$, $j, k \in \mathbb{Z}$; $\langle u, R_k v \rangle = u * \tilde{v}(k)$, $k \in \mathbb{Z}$.

3. J^{th}-stage Framelet Packets on $\mathbb{Z}$.

Theorem 3.1. *Let $L \in \mathbb{N}$. Suppose $u, v^1, v^2, \ldots, v^L \in l(\mathbb{Z})$. Define the matrix of $(u, v^1, v^2, \ldots, v^L)$ by*

$$A(\xi) = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{u}(\xi) & \hat{v^1}(\xi) & \hat{v^2}(\xi) & \ldots & \hat{v^L}(\xi) \\ \hat{u}(\xi+\pi) & \hat{v^1}(\xi+\pi) & \hat{v^2}(\xi+\pi) & \ldots & \hat{v^L}(\xi+\pi) \end{bmatrix}. \tag{6}$$

If

$$A(\xi)A(\xi)^H = I_{2\times 2}, \quad \xi \in [0, \pi), \tag{7}$$

then $B = \{R_{2k}u : k \in \mathbb{Z}\} \cup \{R_{2k}v^l : k \in \mathbb{Z}, l = 1, 2, \ldots, L\}$ is a Parseval frame for $l^2(\mathbb{Z})$, where $A(\xi)^H$ is the conjugate transposed matrix of $A(\xi)$, and $I_{2\times 2}$ is the 2×2 identity matrix.

Proof: It is necessary to prove that for all $z \in l^2(\mathbb{Z})$,

$$\sum_{k\in\mathbb{Z}} |\langle z, R_{2k}u\rangle|^2 + \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}} \left|\langle z, R_{2k}v^l\rangle\right|^2 = \|z\|^2,$$

which is equivalent to

$$\sum_{k\in\mathbb{Z}} \langle z, R_{2k}u\rangle R_{2k}u + \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}} \langle z, R_{2k}v^l\rangle R_{2k}v^l = z$$

or

$$u * U(D(z * \tilde{u})) + \sum_{l=1}^{L} v^l * U\left(D\left(z * \tilde{v^l}\right)\right) = z. \tag{8}$$

To see this, note that by $U(D(z)) = (z + z^*)/2$, then

$$\begin{aligned} &(u * U(D(z * \tilde{u})))^\wedge(\xi) + \sum_{l=1}^{L} \left(v^l * U\left(D\left(z * \tilde{v^l}\right)\right)\right)^\wedge(\xi) \\ &= \hat{u}(\xi)\frac{1}{2}\left[\hat{z}(\xi)\overline{\hat{u}(\xi)} + \hat{z}(\xi+\pi)\overline{\hat{u}(\xi+\pi)}\right] + \sum_{l=1}^{L} \hat{v^l}(\xi)\frac{1}{2}\left[\hat{z}(\xi)\overline{\hat{v^l}(\xi)} + \hat{z}(\xi+\pi)\overline{\hat{v^l}(\xi+\pi)}\right] \\ &= \hat{z}(\xi)\frac{1}{2}\left[|\hat{u}(\xi)|^2 + \sum_{l=1}^{L}\left|\hat{v^l}(\xi)\right|^2\right] + \hat{z}(\xi+\pi)\frac{1}{2}\left[\hat{u}(\xi)\overline{\hat{u}(\xi+\pi)} + \sum_{l=1}^{L}\hat{v^l}(\xi)\overline{\hat{v^l}(\xi+\pi)}\right]. \end{aligned}$$

The condition (7) implies that

$$|\hat{u}(\xi)|^2 + \sum_{l=1}^{L}\left|\hat{v^l}(\xi)\right|^2 = 2, \quad \hat{u}(\xi)\overline{\hat{u}(\xi+\pi)} + \sum_{l=1}^{L}\hat{v^l}(\xi)\overline{\hat{v^l}(\xi+\pi)} = 0,$$

so the last expression reduces to

$$\hat{z}(\xi)\cdot 1 + \hat{z}(\xi+\pi)\cdot 0 = \hat{z}(\xi).$$

By Fourier inversion, this implies Equation (8).

Definition 3.1. *Let $L \in \mathbb{N}$. Suppose $u, v^1, v^2, \ldots, v^L \in l(\mathbb{Z})$. If B is a Parseval frame for $l^2(\mathbb{Z})$, we call B a first-stage Parseval wavelet frame for $l^2(\mathbb{Z})$, and each element in $\{u, v^1, v^2, \ldots, v^L\}$ the mother framelet, where B is defined as in Theorem 3.1.*

Suppose J is a positive integer. For convenience, let $v^0 = u$. Define $w_l^1 = v^l$, $l = 0, 1, \ldots, L$, and, inductively, for $j = 2, 3, \ldots, J$,

$$\omega^j_{n(L+1)+l} = \omega^{j-1}_n * U^{j-1}(v^l), \quad n = 0, 1, \ldots, (L+1)^{j-1} - 1, \quad l = 0, 1, \ldots, L. \tag{9}$$

Theorem 3.2. *Suppose j is a positive integer, $n = 0, 1, \ldots, (L+1)^{j-1}-1$ and $\omega_n^{j-1} \in l(\mathbb{Z})$. Further, suppose $u = v^0, v^1, v^2, \ldots, v^L \in l(\mathbb{Z})$ and the matrix $A(\xi)$ defined as in Equation (6) satisfies Equation (7). Define $\omega^j_{n(L+1)+l}$ as in Equation (9). Then*

$$\sum_{k\in\mathbb{Z}} \left|\left\langle z, R_{2^{j-1}k} w_n^{j-1}\right\rangle\right|^2 = \sum_{l=0}^{L}\sum_{k\in\mathbb{Z}} \left|\left\langle z, R_{2^j k} w^j_{n(L+1)+l}\right\rangle\right|^2, \quad z \in l^2(\mathbb{Z}). \tag{10}$$

Proof: We first claim that

$$\sum_{l=0}^{L} v^l * U\left(D^j\left(z * \widetilde{\omega_n^{j-1}} * U^{j-1}\left(\tilde{v^l}\right)\right)\right) = D^{j-1}\left(z * \widetilde{\omega_n^{j-1}}\right) \tag{11}$$

for all $z \in l^2(\mathbb{Z})$. To see that, note that by $U \circ D(z) = (z + z^*)/2$ and $D^{j-1}(z) * \omega = D^{j-1}\left(z * U^{j-1}(\omega)\right)$, then

$$\begin{aligned}
&\sum_{l=0}^{L}\left(v^l * U\left(D^j\left(z * \widetilde{\omega_n^{j-1}} * U^{j-1}\left(\tilde{v^l}\right)\right)\right)\right)^{\wedge}(\xi)\\
&= \sum_{l=0}^{L}\left(v^l * (U\circ D)\left(D^{j-1}\left(z * \widetilde{\omega_n^{j-1}} * U^{j-1}\left(\tilde{v^l}\right)\right)\right)\right)^{\wedge}(\xi)\\
&= \sum_{l=0}^{L}\left(v^l * (U\circ D)\left(D^{j-1}\left(z * \widetilde{\omega_n^{j-1}}\right) * \tilde{v^l}\right)\right)^{\wedge}(\xi)\\
&= \sum_{l=0}^{L}\left(v^l * \frac{D^{j-1}\left(z * \widetilde{\omega_n^{j-1}}\right) * \tilde{v^l} + \left(D^{j-1}\left(z * \widetilde{\omega_n^{j-1}}\right)\right)^* * \left(\tilde{v^l}\right)^*}{2}\right)^{\wedge}(\xi)\\
&= \frac{1}{2}\left(D^{j-1}\left(z * \widetilde{\omega_n^{j-1}}\right)\right)^{\wedge}(\xi)\sum_{l=0}^{L}\left|v^l(\xi)\right|^2\\
&\quad+\frac{1}{2}\left(D^{j-1}\left(z * \widetilde{\omega_n^{j-1}}\right)\right)^{\wedge}(\xi+\pi)\sum_{l=0}^{L}\hat{v^l}(\xi)\overline{\hat{v^l}(\xi+\pi)}.
\end{aligned}$$

The assumption on $A(\xi)$ implies that

$$\left(D^{j-1}\left(z * \widetilde{\omega_n^{j-1}}\right)\right)^{\wedge}(\xi)\cdot 1 + \left(D^{j-1}\left(z * \widetilde{\omega_n^{j-1}}\right)\right)^{\wedge}(\xi+\pi)\cdot 0 = \left(D^{j-1}\left(z * \widetilde{\omega_n^{j-1}}\right)\right)^{\wedge}(\xi).$$

By Fourier inversion, this implies Equation (11). Hence, using the fact that $U^{j-1}(z*\omega) = U^{j-1}(z) * U^{j-1}(\omega)$ and Equation (11) we have

$$\begin{aligned}
&\sum_{l=0}^{L}\omega^j_{n(L+1)+l} * U^j\left(D^j\left(z * \widetilde{\omega^j_{n(L+1)+l}}\right)\right)\\
&= \sum_{l=0}^{L}\omega_n^{j-1} * U^{j-1}(v^l) * U^j\left(D^j\left(z * \widetilde{\omega_n^{j-1}} * U^{j-1}\left(\tilde{v^l}\right)\right)\right)\\
&= \sum_{l=0}^{L}\omega_n^{j-1} * U^{j-1}(v^l) * U\left(D^j\left(z * \widetilde{\omega_n^{j-1}} * U^{j-1}\left(\tilde{v^l}\right)\right)\right)\\
&= \omega_n^{j-1} * U^{j-1}\left(D^{j-1}\left(z * \widetilde{\omega_n^{j-1}}\right)\right),
\end{aligned}$$

which implies that

$$\sum_{k\in\mathbb{Z}} \left\langle z, R_{2^{j-1}k} w_n^{j-1} \right\rangle R_{2^{j-1}k} w_n^{j-1} = \sum_{l=0}^{L} \sum_{k\in\mathbb{Z}} \left\langle z, R_{2^j k} w^j_{n(L+1)+l} \right\rangle R_{2^j k} w^j_{n(L+1)+l}, \quad z \in l^2(\mathbb{Z}). \tag{12}$$

A simple calculation shows that (12) is equivalent to (10), which is the desired result.

Theorem 3.3. *Suppose $u = v^0, v^1, v^2, \ldots, v^L \in l(\mathbb{Z})$ and the matrix $A(\xi)$ defined as in Equation (6) satisfies Equation (7). Further, define ω_n^j as in Equation (9). Then for any positive integer J,*

$$B_1 := \left\{ R_{2^J k} \omega_n^J : n = 0, 1, \ldots, (L+1)^J - 1,\ k \in \mathbb{Z} \right\} \tag{13}$$

generates a Parseval frame for $l^2(\mathbb{Z})$.

Proof: By Theorem 3.1, $\{R_{2k}u : k \in \mathbb{Z}\} \cup \left\{ R_{2k} v^l : k \in \mathbb{Z}, l = 1, 2, \ldots, L \right\}$ is a Parseval frame for $l^2(\mathbb{Z})$. Hence, for all $z \in l^2(\mathbb{Z})$, we have

$$\|z\|^2 = \sum_{k\in\mathbb{Z}} |\langle z, R_{2k}u\rangle|^2 + \sum_{l=1}^{L}\sum_{k\in\mathbb{Z}} \left|\left\langle z, R_{2k} v^l \right\rangle\right|^2 = \sum_{l=0}^{L}\sum_{k\in\mathbb{Z}} \left|\left\langle z, R_{2k}\omega_l^1 \right\rangle\right|^2.$$

For any $j = 2, 3, \ldots, J$, Theorem 3.2 shows that

$$\sum_{k\in\mathbb{Z}} \left|\left\langle z, R_{2^{j-1}k} w_n^{j-1} \right\rangle\right|^2 = \sum_{l=0}^{L}\sum_{k\in\mathbb{Z}} \left|\left\langle z, R_{2^j k} w^j_{n(L+1)+l} \right\rangle\right|^2, \quad z \in l^2(\mathbb{Z});$$

repeating the argument on $\sum_{k\in\mathbb{Z}} |\langle z, R_{2k}u\rangle|^2$, it follows that

$$\|z\|^2 = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{(L+1)^J-1} \left|\left\langle z, R_{2^J k} w_n^J \right\rangle\right|^2,$$

which concludes the proof.

Definition 3.2. *We call $\left\{\omega_n^J : n = 0, 1, \ldots, (L+1)^J - 1\right\}$ the J-stage basic framelet packets if B_1 defined as in (13) is a Parseval frame for $l^2(\mathbb{Z})$.*

4. **Examples.** We start with the identity

$$\left(\cos^2\left(\frac{\xi}{2}\right) + \sin^2\left(\frac{\xi}{2}\right)\right)^2 = 1 \quad \forall\ \xi. \tag{14}$$

Define

$$|\hat{u}(\xi)|^2 = 2\cos^4\left(\frac{\xi}{2}\right), \quad \left|\hat{v^1}(\xi)\right|^2 = 4\cos^2\left(\frac{\xi}{2}\right)\sin^2\left(\frac{\xi}{2}\right), \quad \left|\hat{v^2}(\xi)\right|^2 = \sin^4\left(\frac{\xi}{2}\right). \tag{15}$$

A simple calculation shows that the matrix $A(\xi)$ of (u, v^1, v^2) satisfies Equation (7). Suppose J is a positive integer. For convenience, let $v^0 = u$. Define $w_l^1 = v^l$, $l = 0, 1, 2$, and, inductively, for $j = 2, 3, \ldots, J$,

$$\omega^j_{3n+l} = \omega_n^{j-1} * U^{j-1}(v^l), \quad n = 0, 1, \ldots, 3^{j-1} - 1,\ l = 0, 1, 2. \tag{16}$$

By using the half-angle trigonometric formulae $2\cos^2\xi = 1 + \cos 2\xi$, $2\sin^2\xi = 1 - \cos 2\xi$ and $2 + 2\cos\xi = \left(1 + e^{i\xi}\right)\left(1 - e^{i\xi}\right)$, we have

$$u(n) = \begin{cases} \sqrt{2}/4, & n = 0, \\ \sqrt{2}/2, & n = 1, \\ \sqrt{2}/4, & n = 2, \\ 0, & \text{otherwise}; \end{cases} \qquad v^1(n) = \begin{cases} 1/2, & n = 0, \\ -1/2, & n = 2, \\ 0, & \text{otherwise}; \end{cases}$$

$$v^2(n) = \begin{cases} \sqrt{2}/4, & n = 0, \\ -1/2, & n = 1, \\ \sqrt{2}/4, & n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Figure 1 shows u, v^1, v^2, restricted to the interval $-10 \le n \le 10$.

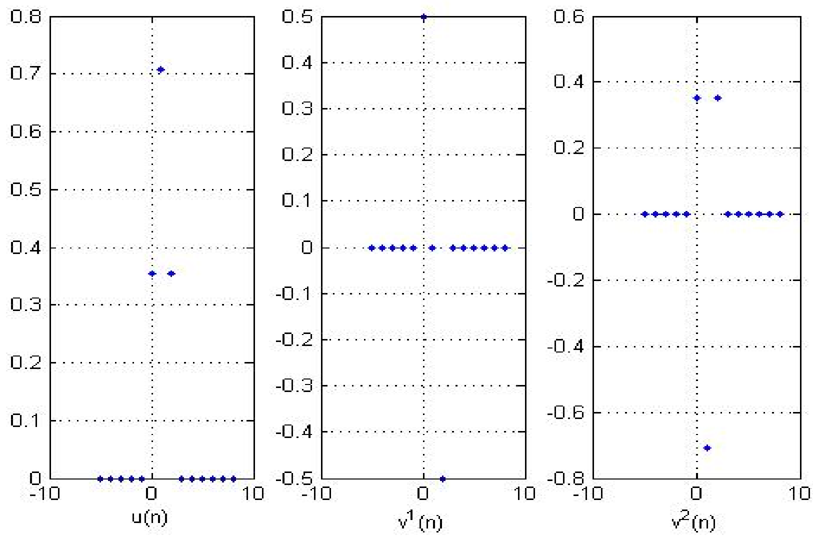

FIGURE 1. u, v^1, v^2

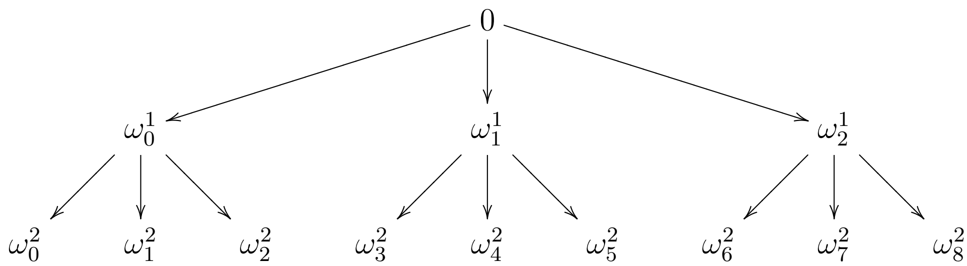

FIGURE 2. 2-stage framelet packet decomposition

By Theorem 3.3, we have that $B_1 := \{R_{2^J k}\omega_n^J : n = 0, 1, \dots, 3^J - 1,\ k \in \mathbb{Z}\}$ generates a Parseval frame for $l^2(\mathbb{Z})$. Figure 2 depicts the 2-stage framelet packet decomposition tree associated with u, v^1 and v^2.

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