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SYNCHRONIZING FRACTIONAL ORDER MODEL UNCERTAIN CHAOTIC SYSTEMS WITH UNKNOWN PARAMETERS AND DISTURBANCES THROUGH ROBUST ADAPTIVE SLIDING MODE CONTROL

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Abstract. *On the basis of Lyapunov stability, an adaptive sliding mode control (ASMC) technique is developed to robustly synchronize fractional order model uncertain chaotic systems with unknown parameters and disturbances. First, the presented sliding mode controller is derived to asymptotically stabilize the error of state. Second, the designed adaptive law guarantees convergence of the unknown parameters. Finally, simulation results demonstrate that ASMC is simultaneously robust against modeling uncertainties, unknown parameters, and external disturbances.*
Keywords: Fractional order, Chaotic system, Adaptive sliding mode control

1. **Introduction.** Chaotic phenomena exist in engineering, biological, chemical, and physical systems; therefore, extensive applications for chaos control and synchronization have been developed in past decades [1-3]. Chaotic behavior is present not only in integer order dynamics but also in fractional order dynamics. Furthermore, some known systems, such as those in secure communication, electronic systems, and control processing, can be elegantly described using fractional calculus rather than using integer order calculus [4-6].

Robust fractional order chaos synchronization is crucial [7]. However, developing a practical robust control strategy is challenging [8]. Chaos systems are sensitive to initial conditions and demonstrate highly nonlinear dynamic behavior. In addition, these systems always induce evitable model uncertainty, unknown parameters, time delay, or disturbances that may deteriorate or even destroy the system synchronization. Many related studies have recently been conducted. Using a backstepping control approach together with sliding mode control (SMC), a robust control technique has been established to overcome model uncertainty in chaotic systems [9,10]. To minimize synchronization error, unknown time delay chaos systems can be approximated using adaptive intelligent networks [11]. An adaptive neural network [12] and adaptive sliding mode [13] were investigated to solve chaotic systems with unknown parameters respectively. An observer-based controller that guarantees that the error of state asymptotically converges to zero was reported in [14,15]. When the assumed linear matrix inequality condition is satisfied, the linear controller eliminates the synchronization error of unknown parameter systems [16].

These advances in robust chaos synchronization are restricted to specific uncertainty conditions; few studies have simultaneously considered multiple and complicated uncertainty conditions. In this paper, a control scheme involving an adaptive control algorithm with a sliding mode technique is presented for synchronizing fractional order model uncertainty chaos systems with unknown parameters and disturbances.

The paper is organized as follows. The problem statement and preliminaries are presented in Section 2. The main results of control scheme are described in Section 3. In Section 4, a simulation example is illustrated to validate the designed controller and then followed by the conclusions in Section 5.

2. **Problem Statement and Preliminaries.** In this section, we use both Riemann-Liouville and Caputo fractional operators [17]. The Riemann-Liouville fractional derivative of order α is defined as

$$D_t^\alpha f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}}d\tau\right] \tag{1}$$

where $m-1<\alpha<m$ and $m \in N$. The variable Γ is the Gamma function.

The Riemann-Liouville fractional integral of order β is defined as

$$J^\beta f(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau)d\tau,\ \beta \in R^+ \tag{2}$$

where Γ is the Gamma function.

The Caputo fractional derivative of order α is defined as

$$D_t^\alpha f(t) = J^{m-\alpha} d^m f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}d\tau \tag{3}$$

where $m-1<\alpha<m$ and $m \in N$. The variable Γ is the Gamma function.

Because the Caputo definition uses the same initial conditions as those in integer order differential equations, it is widely used in engineering applications. Several general properties of the fractional order calculus are used.

Property 1: *For* $\alpha = 0$,

$$D_t^0 f(t) = f(t) \tag{4}$$

Property 2: *For* $\alpha = n$, *where* n *is an integer*,

$$D_t^n f(t) = d^{(n)} f(t) \tag{5}$$

where $d^{(n)}$ is a classical integer order derivative.

Property 3: *Fractional order calculus satisfies the additive law of exponents.*

$$D_t^\alpha D_t^\beta f(t) = D_t^{\alpha+\beta} f(t), \quad D_t^{1-\alpha} D_t^\alpha f(t) = \frac{d}{dt} f(t) \tag{6}$$

Property 4: *Linear operation holds in fractional order calculus.*

$$D_t^\alpha[\lambda^* x(t) + \mu^* y(t)] = \lambda D_t^\alpha x(t) + \mu D_t^\alpha y(t) \tag{7}$$

where λ and μ are real constants.

3. **Main Results.** Consider a class of fractional order uncertainty chaotic with unknown parameters and disturbances described by

$$D_t^\alpha x(t) = F(x)\theta + f(x) + \Delta f(x) + d(t) = Ax + f(x) + \Delta f(x) + d(t) \tag{8}$$

where $0<\alpha<1$, state vector $x = (x_1, x_2, \cdots, x_n)^T$, F is a function matrix, f is a function vector, Δf is the model uncertainty, $d(t)$ is the external disturbance, θ is an unknown parameter vector, and A is an unknown constant matrix.

Let the master system be system (8) and then the slave system is defined as

$$D_t^\alpha y(t) = F(y)\hat{\theta} + f(y) + u = \hat{A}x + f(x) + u \tag{9}$$

where state vector $y = (y_1, y_2, \cdots, y_n)^T$, $\hat{\theta}$ denotes the estimate parameter vector, and $\hat{A}$ denotes the estimated constant matrix.

The following necessary assumptions must be made for synchronizing systems (8) and (9).

Assumption 1: *Suppose the model uncertainty and disturbance can be described as a bound matrix $M(t,y)$. The upper bound of the matrix $M(t,y)$ is γ:*

$$M(t,y) = \Delta f(y) + d(t); \quad \|M(t,y)\| \le \gamma$$

Assumption 2: *Suppose unknown constant matrix $\hat{A}$ and function vector f obey the basis of the boundedness character of chaotic systems. They exist in an upper bound:*

$$\begin{aligned} &\|\hat{A}\| \le \alpha \\ &B_{x,y}(y-x) = f(y) - f(x) \\ &\|B_{x,y}\| \le \beta \end{aligned}$$

Let the synchronization error be $e = y - x$ and the estimated parameter error be $e_\theta = \hat{\theta} - \theta$. The error system is

$$\frac{d^\alpha e}{dt^\alpha} = \left(\hat{A} + B_{x,y}\right) e + F(x)e_\theta + \Delta f(y) + d(t) + u \tag{10}$$

Assumption 3: *The error $e = y - x$ satisfies the Lipschitz conditions [18]. A positive number l exists such that $\|e\| \le l\|s\|$.*

Design sliding surface $s = J_t^{1-\alpha} e$. According to Property 3, we have $\dot{s} = Ds = D_t^\alpha e$.

Suppose that Assumptions 1, 2, and 3 hold, then, the error system (10) is asymptotically stable if and only if the following controller and adaptive laws are chosen.

Controller:

$$u = -ks - \gamma * sign(s) \tag{11}$$

Adaptive law:

$$\dot{k} = \sigma|s| \tag{12}$$

$$\dot{\hat{\theta}} = -F^T(x) sign(s) \tag{13}$$

where σ is an arbitrary positive constant.

Proof: Consider the following Lyapunov candidate function:

$$V = |s| + \frac{e_\theta^T e_\theta}{2} + \frac{(k-k^*)^2}{2\sigma} \tag{14}$$

where positive constant $k^* \ge \|\alpha + \beta\|$.

The time derivative of V is

$$\begin{aligned} \dot{V} &= sign(s)\dot{s} + \dot{e}_\theta^T e_\theta + \tfrac{(k-k^*)\dot{k}}{\sigma} \\ &= sign(s)\left[\left(\hat{A} + B_{x,y}\right) e + F(x)e_\theta + \Delta f(y) + d(t) + u\right] - sign(s)F(x)e_\theta + (k-k^*)|s| \\ &= sign(s)\left[\left(\hat{A} + B_{x,y}\right) e + M(t,y) - \gamma * sign(s)\right] - k^*|s| \\ &\le l(\alpha+\beta)s * sign(s) - k^*|s| \\ &= [l(\alpha+\beta) - k^*] * |s| \le 0 \end{aligned}$$

Because $\dot{V}$ is negative semi definite, all state variables are bounded. Using the Barbalat lemma, if $s, \dot{s} \in L_\infty$, as $t \to \infty$, s approach zero. Consequently, the global synchronization of systems (8) and (9) is completed.

4. **Applications.** The following example is demonstrated to illustrate the effectiveness of the proposed methodology. The Grunwald-Letnikov definition of fractional order calculus is used in the numerical simulation [7].

Let us consider the non-commensurate fractional order model uncertainty Chen system with unknown parameters and disturbances described in [15]:

$$\begin{aligned} &D_t^{\alpha 1} x_1 = a(x_2 - x_1) + d_1(t) \\ &D_t^{\alpha 2} x_2 = -7x_1 + cx_2 - x_1 x_3 + d_2(t) \end{aligned} \tag{15}$$

$$D_t^{\alpha 3} x_3 = x_1 x_2 - b x_3 + \Delta f_3(t, x_1, x_2) + d_3(t)$$

where a, b, c are unknown parameters, $\Delta f(t, X)$ is model uncertainty, and d_i $(i = 1, 2, 3)$ is the disturbance.

When $(a, b, c) = (35, 3, 28)$, $(\alpha 1, \alpha 2, \alpha 3) = (0.95, 0.96, 0.97)$, $\Delta f_3 = 0.1 \sin(t) x_1 x_2$, and disturbance vector $(d_1, d_2, d_3) = (0.3G, 0.3G, 0.3G)$, where G is the Gaussian noise. Figure 1 shows the phase portrait of the fractional order Chen system (15).

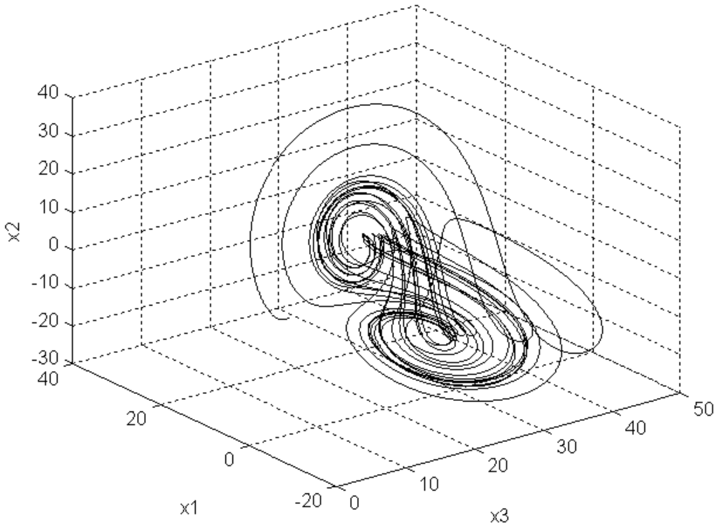

FIGURE 1. Phase portrait of the fractional order Chen system with $(\alpha 1, \alpha 2, \alpha 3) = (0.95, 0.96, 0.97)$

The slave system is described as

$$\begin{aligned} D_t^{\alpha 1} y_1 &= \hat{a}(y_2 - y_1) + u_1 \\ D_t^{\alpha 2} y_2 &= -7y_1 + \hat{c} y_2 - y_1 y_3 + u_2 \\ D_t^{\alpha 3} y_3 &= y_1 y_2 - \hat{b} y_3 + u_3 \end{aligned} \tag{16}$$

where $\left(\hat{a}, \hat{b}, \hat{c}\right)$ are estimations of the unknown parameters (a, b, c), and $(u_1, u_2, u_3)^T$ are the controller vectors.

According to the controller scheme (11), the controllers are

$$u_1 = -ks_1 - \gamma * sign(s_1),\ u_2 = -ks_2 - \gamma * sign(s_2),\ u_3 = -ks_3 - \gamma * sign(s_3)$$

and the adaptive law is represented as

$$\begin{aligned} \dot{k} &= \sigma[|s_1| + |s_2| + |s_3|] \\ \dot{\hat{a}} &= -(x_2 - x_1) sign(s_1) \\ \dot{\hat{b}} &= x_3 sign(s_3) \\ \dot{\hat{c}} &= -x_2 sign(s_2) \end{aligned}$$

where $s_i = J_t^{1-\alpha} e_i$ and $e_i = y_i - x_i$ $(i = 1, 2, 3)$.

In the simulation, the parameters are chosen as $\gamma = 30$, $\sigma = 1$, and $\left(\hat{a}, \hat{b}, \hat{c}\right) = (40, 2, 22)$. Initial conditions of the master and slave system are set as $x(0) = (0.2, 0.5, 0.6)^T$ and $y(0) = (1, -2, 3)^T$, respectively.

The numerical results of the synchronization error between master and slave and the estimated parameters $\left(\hat{a}, \hat{b}, \hat{c}\right)$ of the unknown system are shown in Figure 2. The results

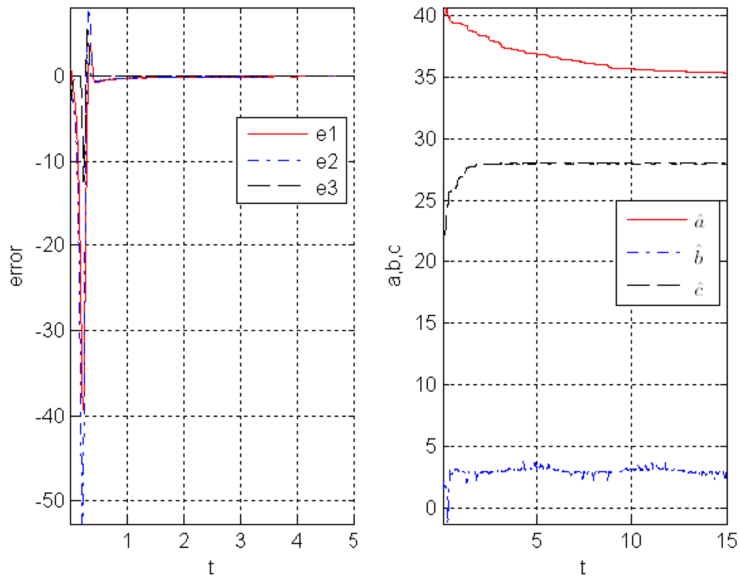

Figure 2. Synchronization errors between the master and slave systems and estimated parameters of the unknown chaos system

demonstrate that the proposed control scheme not only facilitates uncertainty chaotic system synchronization but also estimates the unknown system parameters.

5. **Conclusions.** Practically, evitable uncertainties and disturbances always exist in systems. The parameters of chaos system are always unknown. In this study, a robust adaptive SMC control is proposed to synchronize non-commensurate fractional order model uncertainty chaotic systems with unknown parameters and disturbances. Numerical results show that the merit of the proposed scheme is feasible and that model uncertainty, unknown parameters, and external disturbances of the system are fully accounted for. The future research work will extend control law from 3 control signals (u_1, u_2, u_3) into single control signal u.

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